Chapter 14 Radiation by Moving Charges

14.1 Liénard-Wiechert Potentials and Fields for a Point Charge

- $\vec{A}(\vec{x}) = \frac{4\pi}{c}\int D_r(\vec{x}-\vec{x}\,')\vec{J}(\vec{x}\,')\,\mathrm{d}^4x' \quad \Leftarrow$ no incoming fields

  $D_r(\vec{x}-\vec{x}\,')$: the retarded Green function

$$= 2e\int \vec{U}(\tau)\,\theta[x_0 - r_0(\tau)]\,\delta[(\vec{x}-\vec{r}(\tau))^2]\,\mathrm{d}\tau \quad \Leftarrow \quad \vec{J} = ec\int \vec{U}\,\delta^{(4)}[\vec{x}\,'-\vec{r}]\,\mathrm{d}\tau$$

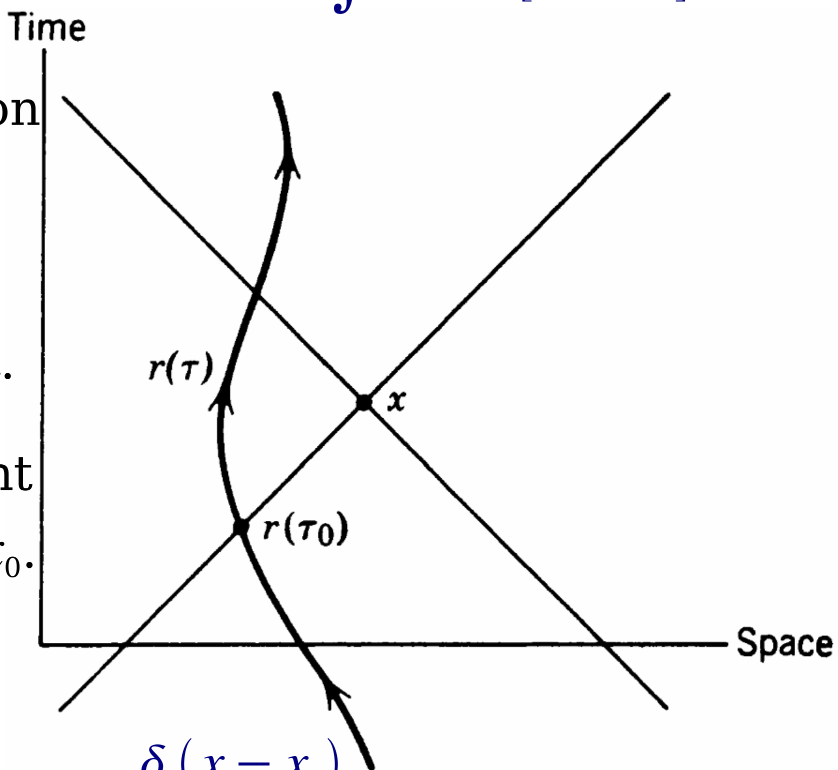
- The integral gives a contribution only

(1) $\tau=\tau_0 \;\Leftarrow [\vec{x}-\vec{r}(\tau_0)]^2 = 0$ the light-cone condition

(2) $x_0 > r_0(\tau_0)$ the retardation requirement

- The Green function is different from 0 only on the backward light cone of the observation point.

- The world line of the particle intersects the light cone at 2 points, one earlier and one later than x_0. The earlier point is the only part of the path that contributes to the fields at x^α.



- $\delta[(\vec{x}-\vec{r}(\tau))^2] = \frac{\delta[\vec{x}-\vec{r}(\tau)]}{2\left|[\vec{x}-\vec{r}(\tau)]\cdot\vec{U}(\tau)\right|} \quad \Leftarrow \quad \delta[f(x)] = \sum \frac{\delta(x-x_i)}{|\mathrm{d}f/\mathrm{d}x|_{x=x_i}}$

$$\Rightarrow \quad \boxed{\vec{A}(\vec{x}) = \frac{e\vec{U}(\tau)}{\vec{U}\cdot[\vec{x}-\vec{r}(\tau)]}\Big|_{\tau=\tau_0}} \qquad \textit{Lienard - Wiechert Potentials}$$

- $\vec{U}\cdot(\vec{x}-\vec{r}) = U_0[x_0 - r_0(\tau_0)] - \mathbf{U}\cdot[\mathbf{x}-\mathbf{r}(\tau_0)]$

$$= \gamma c R - \gamma \mathbf{v}\cdot\mathbf{n} R = \gamma c R (1-\boldsymbol{\beta}\cdot\mathbf{n}) \quad \Leftarrow \quad x_0 - r_0(\tau_0) = |\mathbf{x}-\mathbf{r}(\tau_0)| \equiv R$$

$$\Rightarrow \boxed{\Phi(\mathbf{x},t) = \left[\frac{e}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}\right]_{\text{ret}}, \quad \mathbf{A}(\mathbf{x},t) = \left[\frac{e\boldsymbol{\beta}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}\right]_{\text{ret}}} \Leftarrow$$

$[\;]_{\text{ret}}$: evaluated at the retarded time $\tau_{0,}$ with $r_0(\tau_0) = x_0 - R$

$$\Rightarrow \quad \Phi(\mathbf{x},t) \to \frac{e}{R}, \quad \mathbf{A}(\mathbf{x},t) = \frac{e\,\mathbf{v}}{c\,R} \quad \text{for} \quad \beta \to 0 \quad \text{nonrelativistic motion}$$

- $\partial^\alpha \delta[(\vec{x}-\vec{r}(\tau))^2] = -\dfrac{x^\alpha - r^\alpha}{\vec{U}\cdot(\vec{x}-\vec{r})}\dfrac{d}{d\tau}\delta[(\vec{x}-\vec{r}(\tau))^2] \quad \Leftarrow \quad \partial^\alpha \delta(f) = \partial^\alpha f \dfrac{d\tau}{df}\dfrac{d}{d\tau}\delta(f)$

$$\Rightarrow \quad \partial^\alpha A^\beta = 2e\int U^\beta(\tau)\,\theta[x_0 - r_0(\tau)]\,\partial^\alpha \delta[(\vec{x}-\vec{r}(\tau))^2]\,d\tau \quad ⇦ \quad \partial^\alpha\theta \text{ has no contribution}$$

$$= 2e\int \theta[x_0 - r_0(\tau)]\,\delta[(\vec{x}-\vec{r}(\tau))^2]\frac{d}{d\tau}\frac{(x^\alpha - r^\alpha)U^\beta}{\vec{U}\cdot(\vec{x}-\vec{r})}\,d\tau$$

$$\Rightarrow \boxed{F^{\alpha\beta} = \frac{e}{\vec{U}\cdot(\vec{x}-\vec{r})}\frac{d}{d\tau}\frac{(x^\alpha - r^\alpha)U^\beta - (x^\beta - r^\beta)U^\alpha}{\vec{U}\cdot(\vec{x}-\vec{r})}\Bigg|_{\tau=\tau_0}}$$

- $\vec{U} = (\gamma c, \gamma c \boldsymbol{\beta}) \quad \Rightarrow \quad \dfrac{d}{d\tau}\vec{U} = [c\gamma^4 \boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}, c\gamma^2\dot{\boldsymbol{\beta}} + c\gamma^4(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})\boldsymbol{\beta}] \quad \Leftarrow \quad \dot{\boldsymbol{\beta}} = \dfrac{d}{dt}\boldsymbol{\beta}$

$$\vec{x}-\vec{r} = (R, R\mathbf{n}), \qquad \frac{d}{d\tau}[\vec{U}\cdot(\vec{x}-\vec{r})] = -c^2 + (\vec{x}-\vec{r})\cdot\frac{d}{d\tau}\vec{U}$$

$$\Rightarrow \quad \boxed{\mathbf{E}(\mathbf{x},t) = e\left.\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R^2}\right|_{\text{ret}} + \frac{e}{c}\left.\frac{\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}})}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right|_{\text{ret}}, \quad \mathbf{B} = \mathbf{n}\times\mathbf{E}\big|_{\text{ret}}} \qquad (0)$$

(velocity field) (acceleration field)

- The velocity fields are static fields falling off as R^{-2}, the acceleration fields are radiation fields, **E** & **B** being transverse to the radius vector and varying as R^{-1}.

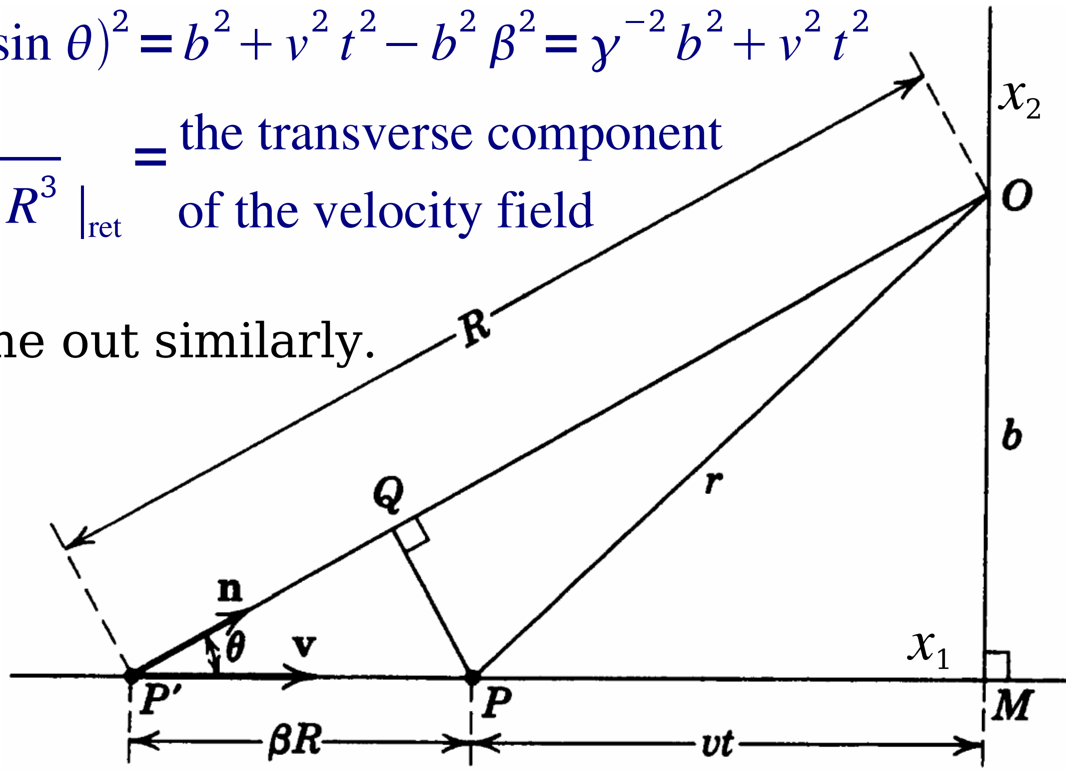
- $\vec{U} = \text{const} \quad \Rightarrow \quad F^{\alpha\beta} = e\,c^2 \left.\frac{(x^\alpha - r^\alpha)U^\beta - (x^\beta - r^\beta)U^\alpha}{[\vec{U}\cdot(\vec{x}-\vec{r})]^3}\right|_{\tau=\tau_0} \quad \Rightarrow \quad$ Sec. 11.10

- $\overline{P'Q} = R\,\beta\cos\theta = \boldsymbol{\beta}\cdot R\,\mathbf{n}, \quad \overline{OQ} = R(1-\boldsymbol{\beta}\cdot\mathbf{n}), \quad b = R\sin\theta$

$$\Rightarrow \quad R^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^2 = r^2 - \overline{PQ}^2 = r^2 - (R\,\beta\sin\theta)^2 = b^2 + v^2t^2 - b^2\beta^2 = \gamma^{-2}b^2 + v^2t^2$$

$$\Rightarrow \quad E_2 = \frac{e\,\gamma\,b}{(b^2+\gamma^2v^2t^2)^{3/2}} = \left.\frac{e\,b}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R^3}\right|_{\text{ret}} = \text{the transverse component of the velocity field}$$

- The other components of **E** and **B** come out similarly.

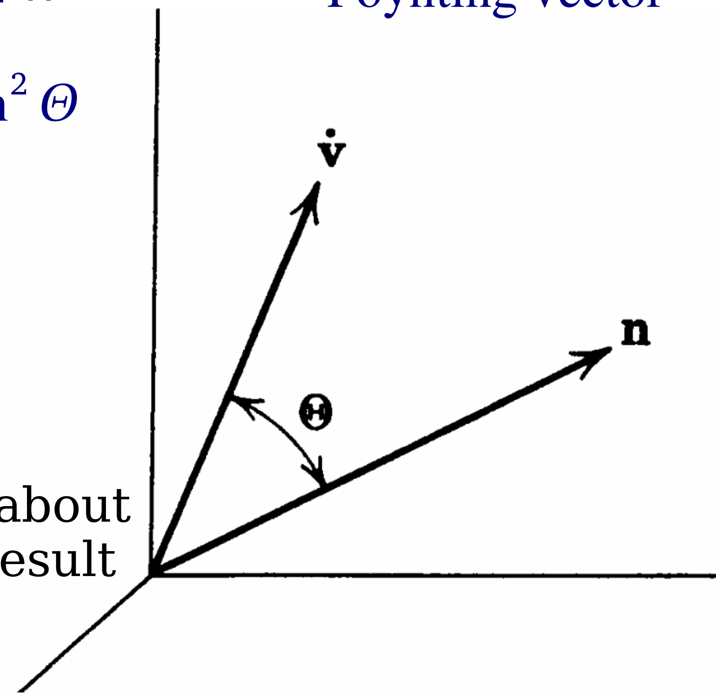


14.2 Total Power Radiated by an Accelerated Charge: Larmor's Formula and Its Relativistic Generalization

- $\beta \ll 1 \quad \Rightarrow \quad \mathbf{E}_a \simeq \frac{e}{c}\left[\frac{\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})}{R}\right]_{\text{ret}} \quad \Rightarrow \quad \mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} \simeq \frac{c}{4\pi}E_a^2\,\mathbf{n} \quad \Leftarrow$ energy flux by Poynting vector

$$\Rightarrow \quad \frac{\mathrm{d}P}{\mathrm{d}\Omega} \simeq \frac{c}{4\pi}R^2 E_a^2 = \frac{e^2}{4\pi c}\left|\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\right|^2 = \frac{e^2}{4\pi c^3}\dot{v}^2\sin^2\Theta$$

$$\Rightarrow \quad \boxed{P = \frac{2}{3}\frac{e^2}{c^3}\dot{v}^2} \quad \Leftarrow$$ Larmor's formula for a nonrelativistic, accelerated charge



the radiation is polarized in the plane of d$\mathbf{v}$/dt and $\mathbf{n}$.

- Larmor's formula can be generalized by arguments about covariance under Lorentz transformations to yield a result that is valid for arbitrary velocities of the charge.

- Radiated EM energy behaves like the 0th component of a 4-vector, so the power is a Lorentz invariant.

- find a Lorentz invariant that involves only $\boldsymbol{\beta}$ and d$\boldsymbol{\beta}$/dt and reduces to Larmor's formula for $\beta \ll 1$, then we have the desired generalization. The result is unique.

- $P = \frac{2}{3}\frac{e^2}{c^3}\dot{v}^2 = \frac{2}{3}\frac{e^2}{m^2c^3}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \quad \Rightarrow \quad P = -\frac{2}{3}\frac{e^2}{m^2c^3}\frac{\mathrm{d}\vec{p}}{\mathrm{d}\tau}\cdot\frac{\mathrm{d}\vec{p}}{\mathrm{d}\tau} \quad (1) \quad \Leftarrow$ generalization

$$-\frac{d\vec{p}}{d\tau}\cdot\frac{d\vec{p}}{d\tau} = \left(\frac{d\mathbf{p}}{d\tau}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2 = \left(\frac{d\mathbf{p}}{d\tau}\right)^2 - \beta^2\left(\frac{dp}{d\tau}\right)^2 \quad \Leftarrow \quad E = \gamma m c^2, \quad \mathbf{p} = \gamma m \mathbf{v}$$

$$dE = m\gamma^3 v\, dv, \quad dp = m\gamma^3 dv$$

$$\Rightarrow \quad P = \frac{2}{3}\frac{e^2}{c}\gamma^6\left[\dot{\boldsymbol{\beta}}^2 - (\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})^2\right] \quad \text{the Lienard result} \quad \Leftarrow \quad dt = \gamma\, d\tau$$

- the expression for radiated power can be used for charged-particle accelerators. Radiation losses are a limiting factor in the maximum practical energy attainable.

- For a given applied force, the radiated power (1) depends inversely on mass^2 of the particle. Consequently these radiative effects are largest for electrons.

- In a linear accelerator the motion is 1d

$$\frac{2}{3}\frac{e^2}{m^2c^3}\left[\left(\frac{dp}{d\tau}\right)^2 - \beta^2\left(\frac{dp}{d\tau}\right)^2\right] \Leftarrow P = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dp}{dt}\right)^2 = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dE}{dx}\right)^2 \quad \Leftarrow \quad \frac{dE}{dp} = \frac{dx}{dt}$$

for linear motion the power radiated depends only on the external forces that determine dE/dx, not on the actual energy or momentum of the particle.

- $$\frac{\text{the radiated power}}{\text{power by external sources}} = \frac{P}{dE/dt} = \frac{2}{3}\frac{e^2}{m^2c^3}\frac{1}{v}\frac{dE}{dx} \rightarrow \frac{2}{3}\frac{e^2/mc^2}{mc^2}\frac{dE}{dx} \quad \text{for} \quad \beta\rightarrow 1$$

- the radiation loss in an electron linear accelerator is unimportant unless the gain in energy is of the order of $mc^2/(e^2/mc^2) \sim 2\times 10^{14}$MeV/m. So radiation losses are negligible in linear accelerators, whether for electrons or heavier particles.

- In circular accelerators the momentum changes rapidly in direction as the particle rotates, but the change in energy per revolution is small

$$\Rightarrow \quad \left|\frac{\mathrm{d}\,\mathbf{p}}{\mathrm{d}\,\tau}\right| = \gamma\,\omega\,|\mathbf{p}| \gg \frac{1}{c}\frac{\mathrm{d}\,E}{\mathrm{d}\,\tau} \quad \Rightarrow \quad P \approx \frac{2}{3}\frac{e^2}{m^2 c^3}\gamma^2\omega^2|\mathbf{p}|^2 = \frac{2}{3}\frac{e^2 c}{\rho^2}\gamma^4\beta^4 \quad \Leftarrow \quad \omega = \frac{c\,\beta}{\rho}$$

$$\Rightarrow \quad \frac{\text{radiative-energy loss}}{\text{revolution}} = \delta\,E \approx \frac{2\,\pi\,\rho}{c\,\beta}P = \frac{4\,\pi}{3}\frac{e^2}{\rho}\gamma^4\beta^3 \rightarrow 10^{-1}\frac{[E\,(\text{GeV})]^4}{\rho\,(\text{meter})} \quad \text{for} \quad \beta \rightarrow 1$$

$$\rho \simeq 1\ \text{meter}\,, \quad E_{\max} \simeq 0.3\,\text{GeV} \quad \Rightarrow \quad \delta\,E_{\max} = 1\,\text{keV}/\text{revolution}$$

This is less than, but not negligible to, the energy gain of a few KVs/turn.

- At higher energies the limitation on available radiofrequency power to overcome the radiation loss becomes a dominant consideration.

- The power radiated in circular electron accelerators can be expressed numerically as

$$P\,(\text{watts}) = 10^6\,\delta\,E\,(\text{Mev})\,J\,(\text{amp})$$

14.3 Angular Distribution of Radiation Emitted by an Accelerated Charge

- For an accelerated charge with $\beta \ll 1$, the radial component of Poynting's vector

$$[\mathbf{S}\cdot\mathbf{n}]_{\text{ret}} \simeq \frac{e^2}{4\pi c}\left|\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right|^2_{\text{ret}} \quad \Leftarrow \quad \mathbf{S} \simeq \frac{c}{4\pi}\mathbf{E}_a\times\mathbf{B}_a = \frac{c}{4\pi}|\mathbf{E}_a|^2\mathbf{n}$$

energy/area/time at an observation point at t of radiation emitted at $t'=t-R(t')/c$.

- Two types of relativistic effect:

(1) the effect of the spatial relationship between $\boldsymbol{\beta}$ & $\dot{\boldsymbol{\beta}}$, which determines the angular distribution.
(2) The relativistic effect from the transformation from the rest frame to the observer's frame and showing itself by the factors $(1-\boldsymbol{\beta}\cdot\mathbf{n})$ in the denominator.

- For ultrarelativistic particles effect (2) dominates the whole angular distribution.

- to calculate the energy radiated during a finite period $[T_1,T_2]$ of acceleration,

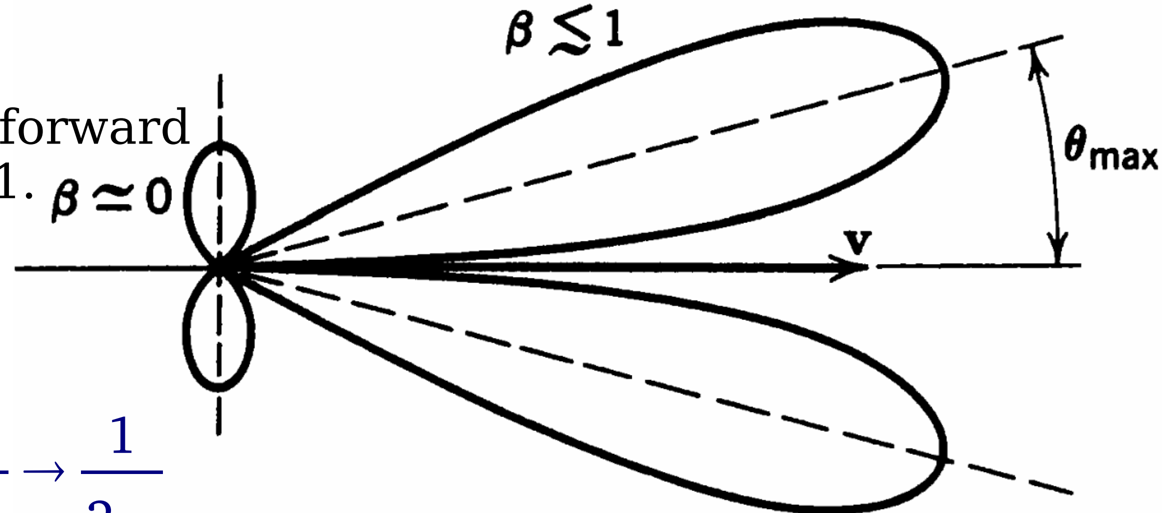
$$E = \int_{t=T_1+R(T_1)/c}^{t=T_2+R(T_2)/c} [\mathbf{S}\cdot\mathbf{n}]_{\text{ret}}\, dt = \int_{t'=T_1}^{t'=T_2} \mathbf{S}\cdot\mathbf{n}\frac{dt}{dt'}dt' \quad \Rightarrow \quad \mathbf{S}\cdot\mathbf{n}\frac{dt}{dt'}: \text{ (power radiated)/area in the charge's time}$$

$$\Rightarrow \quad \frac{\text{power radiated}}{\text{solid angle}} = \frac{dP(t')}{d\Omega} = R^2\,\mathbf{S}\cdot\mathbf{n}\frac{dt}{dt'} = R^2\,\mathbf{S}\cdot\mathbf{n}(1-\boldsymbol{\beta}\cdot\mathbf{n}) = \frac{e^2|\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]|^2}{4\pi c(1-\mathbf{n}\cdot\boldsymbol{\beta})^5}$$

- If the charge is accelerated only for a short time during which $\boldsymbol{\beta}$ & $\dot{\boldsymbol{\beta}}$ basically constant, and the observation point is far away away that $\mathbf{n}$ & R change negligibly during the interval, then the power/(solid angle) is proportional to the angular distribution of the energy radiated.

- $\boldsymbol{\beta} \parallel \dot{\boldsymbol{\beta}}$ for a linear motion $\Rightarrow \dfrac{\mathrm{d}\,P(t')}{\mathrm{d}\,\Omega} = \dfrac{e^2\,\dot{v}^2}{4\,\pi\,c^3}\dfrac{\sin^2\theta}{(1-\beta\cos\theta)^5} \Leftarrow \cos\theta = \mathbf{n}\cdot\hat{\boldsymbol{\beta}}$

$\Rightarrow$ Larmor's result for $\beta \ll 1$



- the angular distribution is tipped forward and increases in magnitude for $\beta \rightarrow 1$.

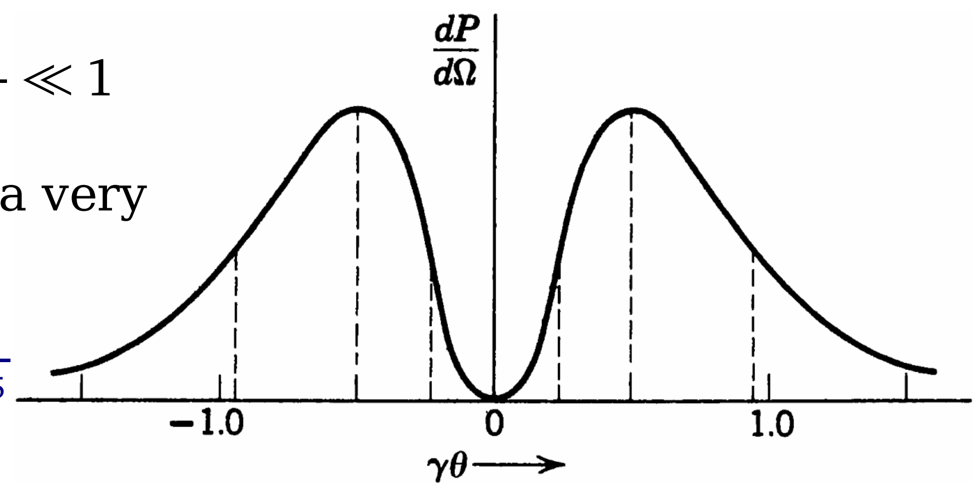
$$\beta \rightarrow 1 \quad \Rightarrow \quad \left.\frac{\mathrm{d}\,P}{\mathrm{d}\,\Omega}\right|_{\max} \propto \gamma^8$$

$$\theta_{\max} = \cos^{-1}\frac{\sqrt{15\,\beta^2+1}-1}{3\,\beta} \rightarrow \frac{1}{2\,\gamma}$$

- For β=0.5, corresponding to electrons of ~80 keV kinetic energy, $\theta_{\max}$=38.2°.

- For relativistic particles, $\theta_{\max} \sim \dfrac{\text{rest energy}}{\text{total energy}} \ll 1$

and the angular distribution is confined to a very narrow cone in the direction of motion.



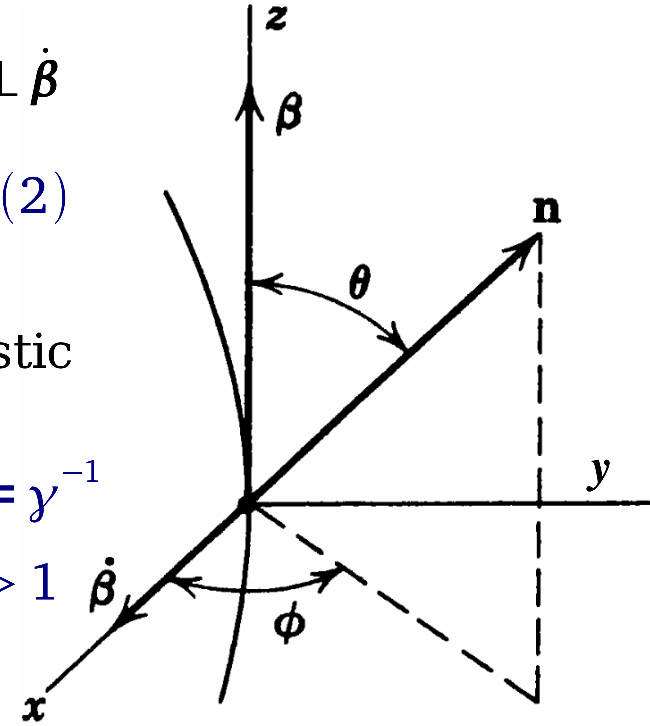
$$\theta \rightarrow 0 \quad \Rightarrow \quad \frac{\mathrm{d}\,P(t')}{\mathrm{d}\,\Omega} \simeq \frac{8}{\pi}\frac{e^2\,\dot{v}^2}{c^3}\gamma^8\frac{\gamma^2\,\theta^2}{(1+\gamma^2\,\theta^2)^5}$$

- The peak occurs at $\gamma\theta=\pm 1/2$, the half-power points at $\gamma\theta=\pm 0.23$ & $\gamma\theta=\pm 0.91$.
- The rms angle of radiation in the relativistic limit $\theta_{\text{rms}} \equiv \sqrt{\langle \theta^2 \rangle} = \dfrac{1}{\gamma} = \dfrac{m c^2}{E}$

typical of the relativistic radiation patterns, regardless of the angle of $\boldsymbol{\beta}$ & $\dot{\boldsymbol{\beta}}$.

- The total power $P_{\text{linear}}(t') = \int \dfrac{d P(t')}{d\Omega} d\Omega = \dfrac{2}{3} \dfrac{e^2}{c^3} \dot{v}^2 \gamma^6 \quad \Rightarrow \quad \text{the Lienard result}$
- for a charge in instantaneously circular motion $\Rightarrow \quad \boldsymbol{\beta} \perp \dot{\boldsymbol{\beta}}$

$$\Rightarrow \quad \frac{d P(t')}{d\Omega} = \frac{e^2}{4\pi c^3} \frac{\dot{v}^2}{(1-\beta\cos\theta)^3} \left[1 - \frac{\sin^2\theta \cos^2\phi}{\gamma^2 (1-\beta\cos\theta)^2} \right] \quad (2)$$



- In the relativistic limit, the same characteristic relativistic peaking at forward angles is present.

$$\frac{d P(t')}{d\Omega} = \frac{2}{\pi} \frac{e^2}{c^3} \gamma^6 \frac{\dot{v}^2}{(1+\gamma^2\theta^2)^3} \left[1 - \frac{4\gamma^2\theta^2\cos^2\phi}{(1+\gamma^2\theta^2)^2} \right] \quad \& \quad \theta_{\text{rms}} = \gamma^{-1} \quad \text{for} \quad \gamma \gg 1$$

- The total power $P_{\text{circular}}(t') = \dfrac{2}{3} \dfrac{e^2}{c^3} \dot{v}^2 \gamma^4$
- For circular motion $\dot{\mathbf{p}} = \gamma m \dot{\mathbf{v}} = \mathbf{F} \quad \Rightarrow \quad P_{\text{circular}}(t') = \dfrac{2}{3} \dfrac{e^2}{m^2 c^3} \gamma^2 \left(\dfrac{d\mathbf{p}}{dt}\right)^2 = \gamma^2 P_{\text{linear}}(t')$
- for a given magnitude of applied force the radiation emitted with a transverse acceleration is a factor of γ^2 larger than with a parallel acceleration.

14.4 Radiation Emitted by a Charge in Arbitrary, Extremely Relativistic Motion

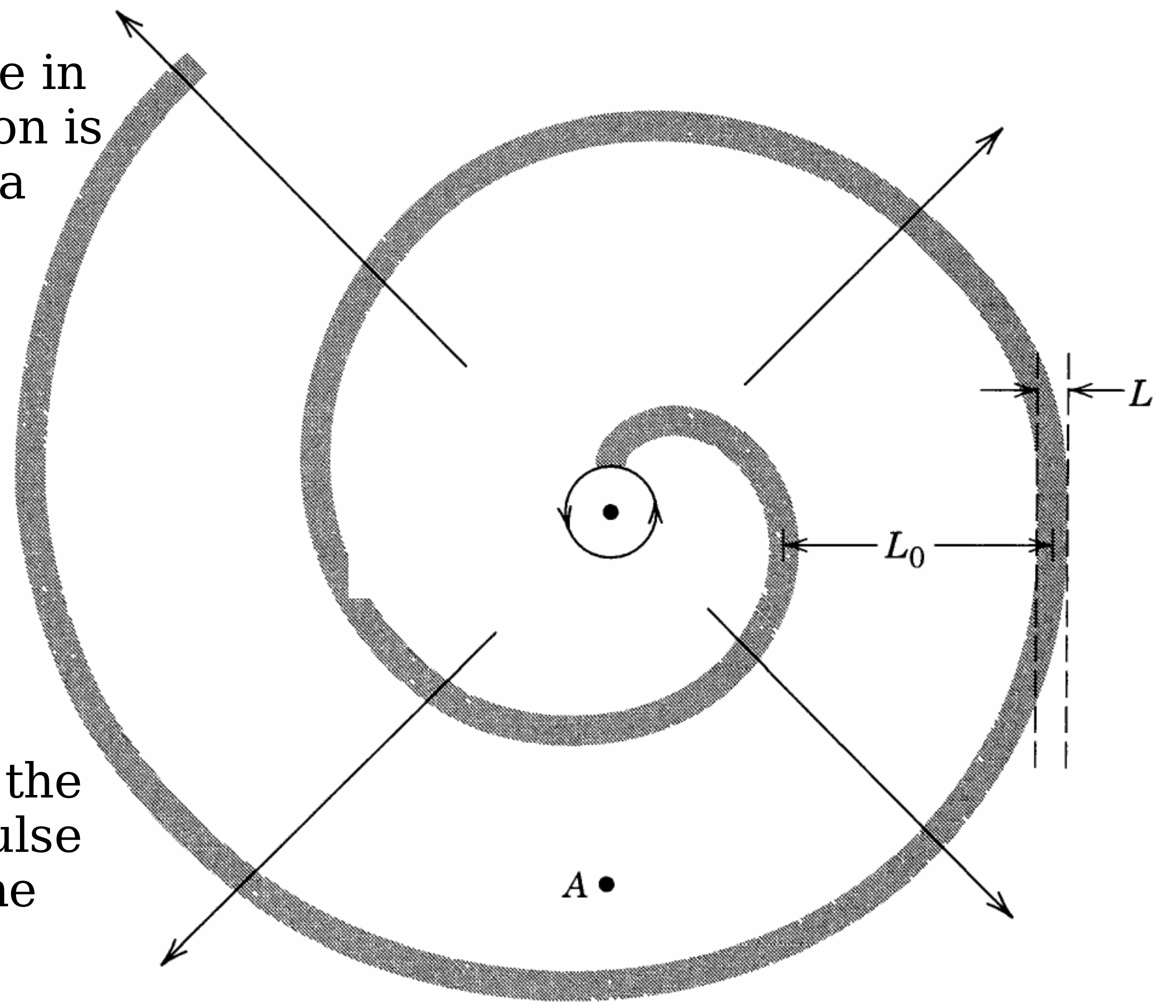
- In the case the radiation can be thought of as a coherent superposition of contributions coming from the components of acceleration $\parallel$ & $\perp$ to the velocity.

- neglect the $\parallel$-component part and approximate the radiation intensity with the $\perp$-component part alone because the radiation from the $\parallel$-component part is of order γ^{-2} compared to that from the $\perp$-component part.

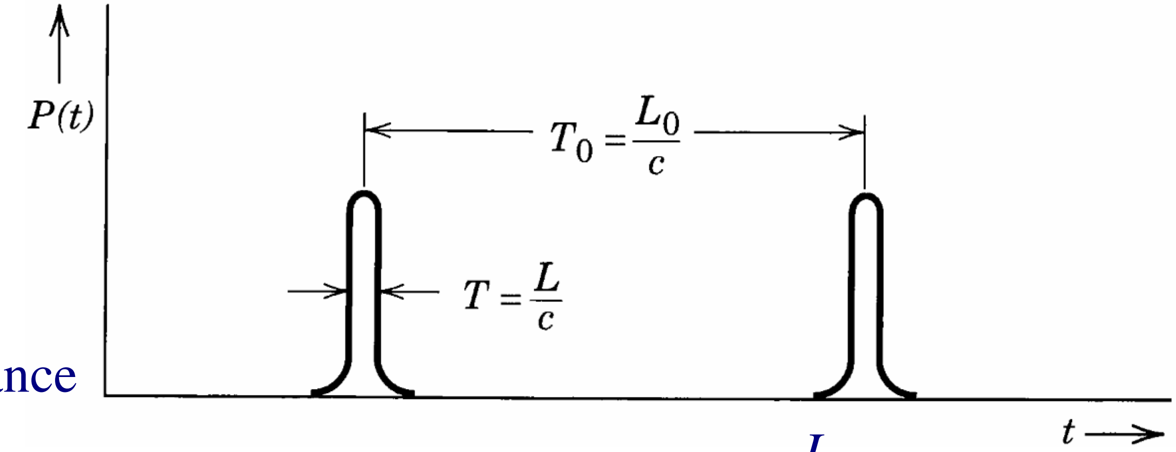
- the radiation by a charged particle in arbitrary, extreme relativistic motion is approximately the same as that by a particle moving instantaneously along the arc of a circular path of radius of curvature

$$\rho = \frac{v^2}{\dot{v}_\perp} \simeq \frac{c^2}{\dot{v}_\perp} \quad \Rightarrow \quad \frac{dP(t')}{d\Omega} = (2)$$

a narrow cone or searchlight beam of radiation directed along the velocity vector of the charge.



- For a particle in arbitrary motion the observer will detect a short-time pulse (or a succession of such bursts if the particle is in periodic motion).



- $\Delta\theta \sim \frac{1}{\gamma} \Rightarrow d \sim \frac{\rho}{\gamma} \Rightarrow \Delta t \sim \frac{\rho}{\gamma v}$

$\Rightarrow D = c\,\Delta t \sim \frac{\rho}{\gamma\beta}$ pulse front's travelling distance

$\Rightarrow L = D - d = \frac{\rho}{\gamma\beta} - \frac{\rho}{\gamma} \simeq \frac{\rho}{2\gamma^3} \Leftarrow$ the length of the pulse $\Rightarrow T = \frac{L}{c}$

- By analyzing the wave trains it implies that the spectrum of the radiation will contain appreciable frequency components up to a critical frequency

$$\omega_c \sim \frac{c}{L} \sim \gamma^3 \omega_0 \quad (3) \quad \Leftarrow \quad \omega_0 = \frac{c}{\rho} \text{ the fundamental frequency}$$

- a relativistic particle emits a broad spectrum of frequencies, up to γ^3 times the fundamental frequency.

- 200 MeV synchrotron $\Rightarrow \gamma_{max} = 400,\ \omega_0 \simeq 3\times 10^8\ \mathrm{s}^{-1} \Rightarrow \omega_c \sim 2\times 10^{16}\ \mathrm{s}^{-1},\ \lambda_c \sim 10^3$ Å
- 10 GeV machine $\Rightarrow \gamma_{max} = 20000,\ \omega_0 \simeq 3\times 10^6\ \mathrm{s}^{-1} \Rightarrow \omega_c \sim 2.4\times 10^{19}\ \mathrm{s}^{-1} \Rightarrow$ 16 keV x-ray

14.5 Distribution in Frequency and Angle of Energy Radiated by Accelerated Charges: Basic Results

- For relativistic motion the radiated energy is over a wide range of frequencies. The frequency spectrum can be analyzed precisely & quantitatively by the use of Parseval's theorem of Fourier analysis.

- **Parseval's theorem**: the sum/integral of the square of a function is equal to the sum/integral of the square of its transform, ie, the Fourier transform is unitary.

- $$\frac{dP(t)}{d\Omega} = |\mathbf{A}(t)|^2 \quad \Leftarrow \quad \mathbf{A}(t) = \sqrt{\frac{c}{4\pi}}\,[\Re(\mathbf{E})]_{\mathrm{ret}} \quad \Leftarrow \quad \text{in the observer's time}$$

$$A(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{A}(t)\, e^{i\omega t}\, dt \quad \Leftrightarrow \quad A(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{A}(\omega)\, e^{-i\omega t}\, d\omega$$

$$\Rightarrow \quad \frac{dW}{d\Omega} = \int \frac{dP(t)}{d\Omega}\, dt = \int_{-\infty}^{\infty} |\mathbf{A}(t)|^2\, dt \qquad \Leftarrow \quad \delta(\omega - \omega') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega-\omega')t}\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{A}^*(\omega')\cdot\mathbf{A}(\omega')\, e^{i(\omega'-\omega)t}\, d\omega'\, d\omega\, dt = \int_{-\infty}^{\infty} |\mathbf{A}(\omega)|^2\, d\omega$$

$$= \int_0^{\infty} \frac{d^2 I(\omega,\mathbf{n})}{d\omega\, d\Omega}\, d\omega \quad \Leftarrow \quad \frac{d^2 I(\omega,\mathbf{n})}{d\omega\, d\Omega} = |\mathbf{A}(\omega)|^2 + |\mathbf{A}(-\omega)|^2$$

$$\Rightarrow \quad \frac{d^2 I(\omega,\mathbf{n})}{d\omega\, d\Omega} = 2\,|\mathbf{A}(\omega)|^2 \quad \text{if} \quad \mathbf{A}(t) \in \mathbb{R} \quad \Leftarrow \quad \mathbf{A}(-\omega) = \mathbf{A}^*(\omega)$$

- $$\mathbf{A}(\omega) = \frac{e}{\sqrt{8\pi^2 c}} \int_{-\infty}^{\infty} \left[\frac{\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}})}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3} \right]_{\text{ret}} e^{i\omega t}\,\mathrm{d}t \quad \text{for an accelerated charge} \quad \Leftarrow \quad (0)$$

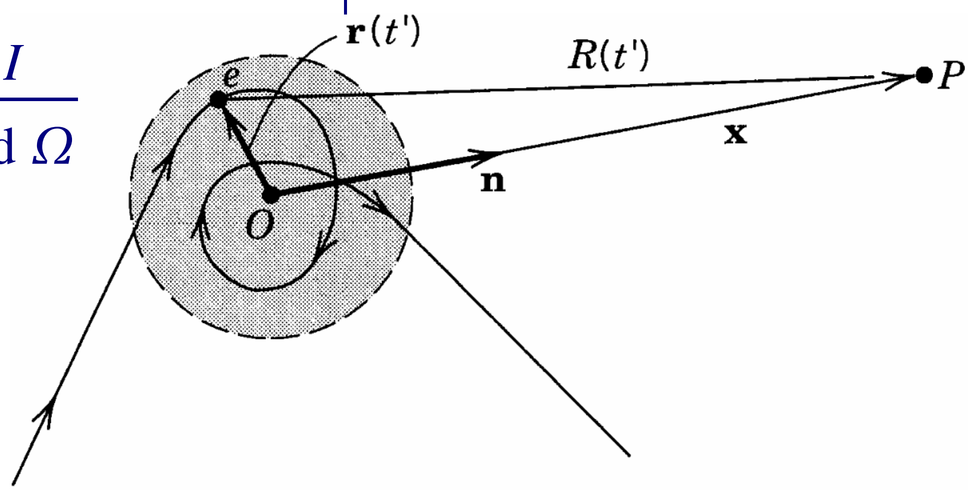
$$= \frac{e}{\sqrt{8\pi^2 c}} \int_{-\infty}^{\infty} \frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^2} e^{i\omega[t'+R(t')/c]}\,\mathrm{d}t' \quad \Leftarrow \quad t = t' + \frac{R(t')}{c}$$

$$= \frac{e}{\sqrt{8\pi^2 c}} e^{i\omega x/c} \int_{-\infty}^{\infty} \frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^2} e^{i\omega[t-\mathbf{n}\cdot\mathbf{r}(t)/c]}\,\mathrm{d}t \quad \Leftarrow \quad R(t') \approx x - \mathbf{n}\cdot\mathbf{r}(t')$$

$$\Rightarrow \quad \frac{\mathrm{d}^2 I}{\mathrm{d}\omega\,\mathrm{d}\Omega} = \frac{e^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} \frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^2} e^{i\omega[t-\mathbf{n}\cdot\mathbf{r}(t)/c]}\,\mathrm{d}t \right|^2$$

given $\mathbf{r}(t) \quad \Rightarrow \quad \boldsymbol{\beta}(t)$ & $\dot{\boldsymbol{\beta}}(t) \quad \Rightarrow \quad \dfrac{\mathrm{d}^2 I}{\mathrm{d}\omega\,\mathrm{d}\Omega}$

$$\frac{\mathrm{d}^2 I}{\mathrm{d}\omega\,\mathrm{d}\Omega} = 2\sum |\mathbf{A}_j(\omega)|^2 \quad \text{for many particle}$$



- $$\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^2} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})}{1-\mathbf{n}\cdot\boldsymbol{\beta}}$$

$$\Rightarrow \quad \frac{\mathrm{d}^2 I}{\mathrm{d}\omega\,\mathrm{d}\Omega} = \frac{e^2\omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\, e^{i\omega[t-\mathbf{n}\cdot\mathbf{r}(t)/c]}\,\mathrm{d}t \right|^2 \qquad (4)$$

- (4) is correct in all circumstances. For the acceleration being different from zero for $T_1 \le t \le T_2$, by adding & subtracting the integrals over the times for v=const, (3) will give right answer.

- In processes like beta decay, involving the almost instantaneous halting or setting in motion of charges, extra care must be taken to specify each particle's velocity as a physically sensible function of time.

- the polarization of the radiation is given by the direction of the vector integral in each. The intensity of radiation of a fixed polarization can be obtained by the scalar product of the unit polarization vector with the vector integral.

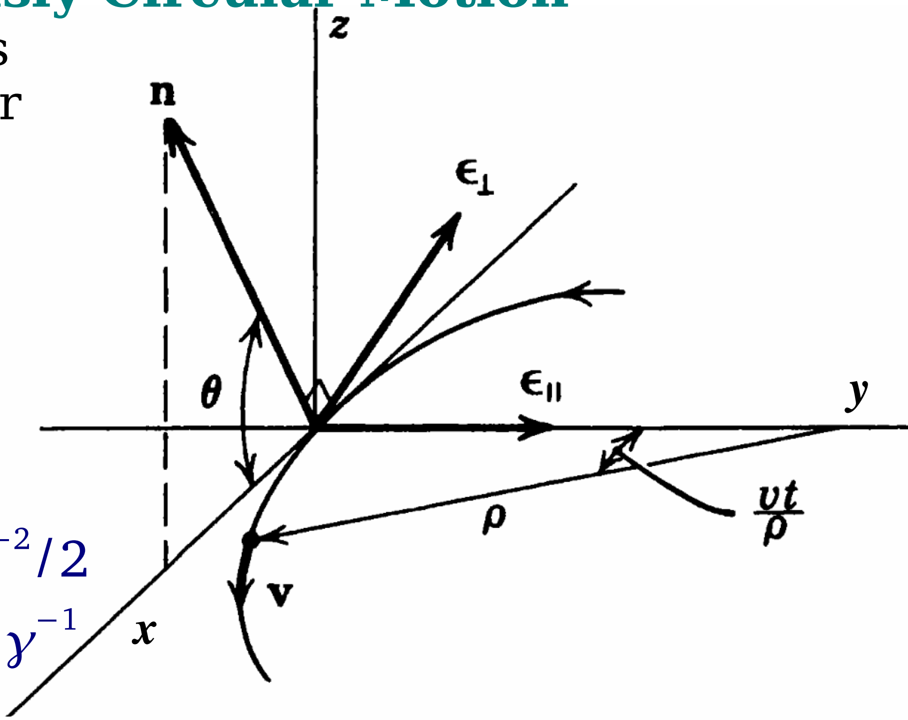
- For a number of charges

$$e\,\boldsymbol{\beta}\,e^{-i\omega\,\mathbf{n}\cdot\mathbf{r}(t)/c} \to \sum_{j=1}^{N} e_j\,\boldsymbol{\beta}_j\,e^{-i\omega\,\mathbf{n}\cdot\mathbf{r}_j(t)/c} \to \frac{1}{c}\int \boldsymbol{J}(\mathbf{x},t)\,e^{-i\omega\,\mathbf{n}\cdot\mathbf{x}/c}\,\mathrm{d}^3x$$

$$\Rightarrow \quad \frac{\mathrm{d}^2 I}{\mathrm{d}\omega\,\mathrm{d}\Omega} = \frac{\omega^2}{4\pi^2 c^3}\left|\int\int \mathbf{n}\times[\mathbf{n}\times\boldsymbol{J}(\mathbf{x},t)]\,e^{i\omega(t-\mathbf{n}\cdot\mathbf{x}/c)}\,\mathrm{d}^3x\,\mathrm{d}t\right|^2$$

a result that can be obtained from the direct solution of the inhomogeneous wave eqn for the vector potential.

14.6 Frequency Spectrum of Radiation Emitted by a Relativistic Charged Particle in Instantaneously Circular Motion



- If the duration of the pulse is very short, it is necessary to know the velocity & position over only a small arc of the trajectory.

- $$\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) = \beta\left[\boldsymbol{\epsilon}_\perp \cos\frac{v\,t}{\rho}\sin\theta - \boldsymbol{\epsilon}_\parallel \sin\frac{v\,t}{\rho}\right]$$

$$1 - \frac{\mathbf{n}\cdot\mathbf{r}(t)}{c} = t - \frac{\rho}{c}\sin\frac{v\,t}{\rho}\cos\theta$$

$$\simeq \frac{1+\gamma^2\theta^2}{2\gamma^2}t + \frac{c^2}{6\rho^2}t^3 \quad\Leftarrow\quad \begin{array}{l}\beta \to 1-\gamma^{-2}/2\\ \theta \leftarrow \theta_{\rm rms} = \gamma^{-1}\end{array}$$

$$\Rightarrow\quad \frac{d^2 I}{d\omega\, d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\boldsymbol{\epsilon}_\perp A_\perp(\omega) - \boldsymbol{\epsilon}_\parallel A_\parallel(\omega)\right|^2 \quad\Leftarrow\quad (3)$$

where

$$A_\parallel(\omega) \simeq \frac{c}{\rho}\int_{-\infty}^{\infty} t\, e^{i\omega\left[\frac{1+\gamma^2\theta^2}{2\gamma^2}t + \frac{c^2 t^3}{6\rho^2}\right]} dt = \frac{\rho}{c}(\gamma^{-2}+\theta^2)\int_{-\infty}^{\infty} x\, e^{i\xi\frac{3x+x^3}{2}} dx$$

$$A_\perp(\omega) \simeq \theta\int_{-\infty}^{\infty} e^{i\omega\left[\frac{1+\gamma^2\theta^2}{2\gamma^2}t + \frac{c^2 t^3}{6\rho^2}\right]} dt = \frac{\rho}{c}\theta\sqrt{\gamma^{-2}+\theta^2}\int_{-\infty}^{\infty} e^{i\xi\frac{3x+x^3}{2}} dx$$

where $x = \dfrac{c\,t}{\rho\sqrt{\gamma^{-2}+\theta^2}}, \quad \xi = \dfrac{\omega\rho}{3c}(\gamma^{-2}+\theta^2)^{3/2}$

$$\int_0^\infty x \sin \frac{\xi(3x + x^3)}{2} \, dx = \frac{1}{\sqrt{3}} K_{2/3}(\xi), \quad \int_0^\infty \cos \frac{\xi(3x + x^3)}{2} \, dx = \frac{1}{\sqrt{3}} K_{1/3}(\xi)$$

$$\Rightarrow \quad \frac{d^2 I}{d\omega \, d\Omega} = \frac{e^2 \omega^2 \rho^2}{3\pi^2 c^3} \frac{(1+\gamma^2\theta^2)^2}{\gamma^4} \left[K^2_{2/3}(\xi) + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2} K^2_{1/3}(\xi) \right] \quad (5)$$

radiation polarized || the plane of the orbit — radiation polarized ⊥ the plane of the orbit

$$\Rightarrow \quad \frac{dI}{d\Omega} = \int_0^\infty \frac{d^2 I}{d\omega \, d\Omega} d\omega = \frac{7}{16} \frac{e^2}{\rho} \frac{\gamma^5}{(1+\gamma^2\theta^2)^{5/2}} \left[1 + \frac{5}{7} \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2} \right] \quad \Leftarrow \quad (2)$$

$\Rightarrow \quad I = I_{\parallel} + I_{\perp} \quad \Leftarrow \quad I_{\parallel} \approx 7 I_{\perp} \quad \Rightarrow$ The radiation from a relativistically moving charge is very strongly polarized in the plane of motion.

- $I \to 0$ as $\xi \gg 1 \Leftarrow$ large θ :the radiation is largely confined to the plane of the motion, being more confined the higher the frequency relative to c/ρ.

- If ω gets too large, ξ will be large at *all* angles. Then there will be negligible total energy emitted at that frequency.

- critical frequency: $\omega_c = \frac{3}{2}\gamma^3 \frac{c}{\rho} = \frac{3}{2}\left(\frac{E}{mc^2}\right)^3 \frac{c}{\rho} \simeq (4) \quad \Leftarrow \quad \xi(\omega_c, \theta = 0) = \frac{1}{2}$

$$\Rightarrow \quad \omega_c = n_c \omega_0 \quad \Rightarrow \quad n_c = 3\gamma^3/2 \quad \Leftarrow \quad \omega_0 = c/\rho$$

critical harmonic frequency — harmonic number — fundamental harmonic frequency

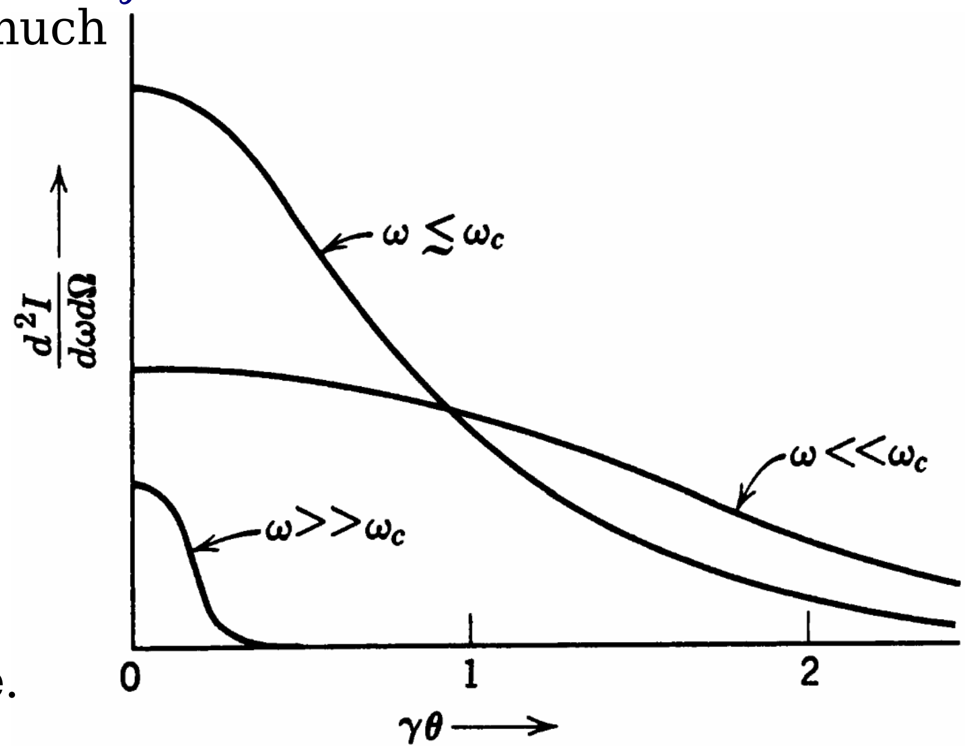
- $$\left.\frac{d^2 I}{d\omega\, d\Omega}\right|_{\theta=0} = \begin{cases} \frac{e^2}{c}\left[\frac{\Gamma(3/2)}{\pi}\right]^2 \left(\frac{3}{4}\right)^{1/3} \left(\frac{\omega\rho}{c}\right)^{2/3} & \text{for} \quad \omega \ll \omega_c \\ \frac{3}{4\pi}\frac{e^2}{c}\gamma^2 \frac{\omega}{\omega_c} e^{-\omega/\omega_c} & \text{for} \quad \omega \gg \omega_c \end{cases}$$

the spectrum at $\theta=0$ increases with frequency as $\omega^{2/3}$ below the critical frequency, reaches a maximum near ω_c, drops exponentially to zero above that frequency.

- Estimate the spread in angle at a fixed frequency by finding $\theta_c \Leftarrow \xi(\theta_c) \simeq \xi(0)+1$

$$\omega \ll \omega_c \quad \Rightarrow \quad \xi(\theta_c) \simeq 1 \quad \Rightarrow \quad \theta_c \simeq \left(\frac{3c}{\omega\rho}\right)^{1/3} = \frac{1}{\gamma}\left(\frac{2\omega_c}{\omega}\right)^{1/3} = \left(\frac{2\omega_c}{\omega}\right)^{1/3} \theta_{\text{rms}} > \theta_{\text{rms}}$$

the low-freq. components are emitted at much wider angles than the average.



- $\omega > \omega_c \quad \Rightarrow \quad \xi(\theta_c) \gg 1$

$$\Rightarrow \quad \frac{d^2 I}{d\omega\, d\Omega} \simeq \left.\frac{d^2 I}{d\omega\, d\Omega}\right|_{\theta=0} e^{-3\omega\gamma^2\theta^2/2\omega_c}$$

$$\Rightarrow \quad \theta_c \simeq \frac{1}{\gamma}\sqrt{\frac{2\omega_c}{3\omega}} \quad \Leftarrow \quad \frac{3\omega\gamma^2\theta_c^2}{2\omega_c} = 1$$

the high-freq. components are within an angular range much smaller than average.

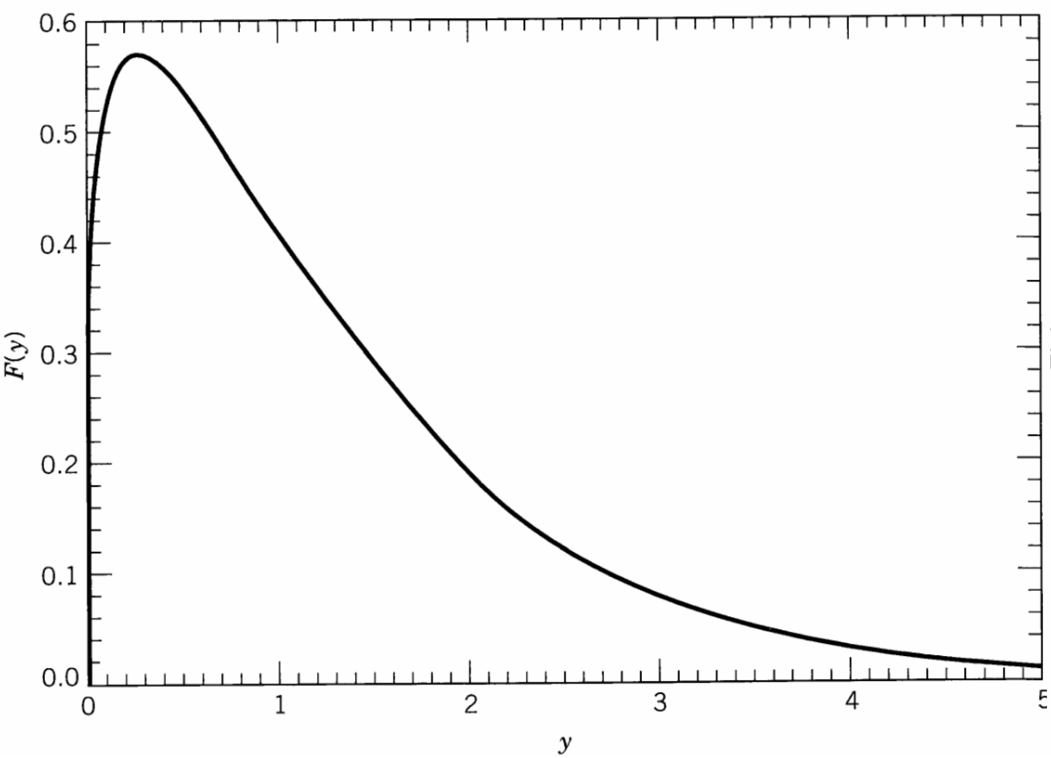
- for the low-frequency range $\omega \ll \omega_c$

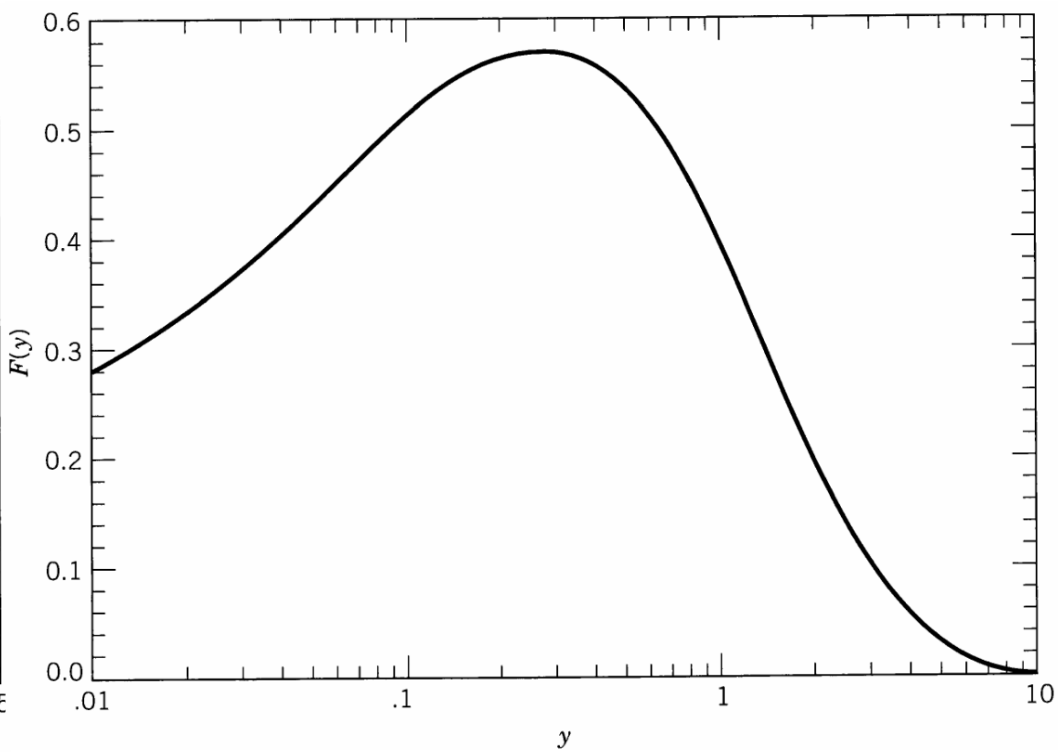
$$\frac{dI}{d\omega} = 2\pi \int_{-\pi/2}^{\pi/2} \frac{d^2 I}{d\omega\, d\Omega}\, d\sin\theta \simeq 2\pi \int_{-\infty}^{\infty} \frac{d^2 I}{d\omega\, d\Omega}\, d\theta \sim 2\pi\, \theta_c \left.\frac{d^2 I}{d\omega\, d\Omega}\right|_{\theta=0} \sim \frac{e^2}{c}\left(\frac{\omega\rho}{c}\right)^{\frac{1}{3}}$$

the spectrum increases as $\omega^{1/3}$, and is very broad, flat at frequencies below ω_c.

- for the high-frequency range $\omega \gg \omega_c$ $\Rightarrow$ $$\frac{dI}{d\omega} \simeq \frac{e^2}{c}\gamma\sqrt{\frac{3\pi}{2}\frac{\omega}{\omega_c}}\, e^{-\omega/\omega_c}$$

- A proper integration gives $$\frac{dI}{d\omega} = \sqrt{3}\,\frac{e^2}{c}\gamma\frac{\omega}{\omega_c}\int_{\omega/\omega_c}^{\infty} K_{5/3}(x)\, dx \quad (6)$$





- The radiation represented by (5) & (6) is called *synchrotron radiation*.

- For periodic circular motion the spectrum is actually discrete, being composed of frequencies that are integral multiples of the fundamental frequency $\omega_0 = c/\rho$.

- Since the charged particle repeats its motion at a rate of $c/2\pi\rho$ rev/sec, it is convenient to talk about the angular distribution of power radiated into the nth multiple of ω_0 instead of the energy radiated/frequency interval/particle.

$$\frac{dP_n}{d\Omega} = \frac{1}{2\pi}\frac{c^2}{\rho^2}\frac{d^2 I}{d\omega\, d\Omega}\Big|_{\omega = n\omega_0}, \qquad P_n = \frac{1}{2\pi}\frac{c^2}{\rho^2}\frac{dI}{d\omega}\Big|_{\omega = n\omega_0}$$

- Due to the broad frequency distribution covering the visible, UV, x-ray regions, synchrotron radiation is a useful tool for studies in condensed matter & biology.

- electrons in the Crab nebula with energies ranging up to 10^{13} eV are emitting synchrotron radiation while moving in circular or helical orbits in a $\mathbf{B} \sim 10^{-4}$ gauss.

- The radio emission at $\sim 10^3$ MHz from Jupiter comes from energetic electrons trapped in Van Allen belts at distances up to 100 radii from Jupiter's surface.

- $B \sim 1$ gauss, $\quad E_e \sim 5$ MeV $\quad \Rightarrow \quad \rho \sim 100$ - 200 meters, $\quad \omega_0 \sim 2 \times 10^6$/s

  $\Rightarrow \quad 10^3$ significant harmonics radiated

- (number of photons)/frequency is to divide the intensity distribution by $\hbar\omega$

$$\frac{dN}{d(\omega/\omega_c)} = \frac{I}{\hbar\,\omega_c}\,\frac{9\sqrt{3}}{8\pi}\int_{\omega/\omega_c}^{\infty} K_{5/3}(x)\,dx \quad \Leftarrow \quad I = \frac{4\pi\,e^2\gamma^4}{3\,\rho} \quad \text{total energy radiated per revolution}$$

$$\Rightarrow \quad \frac{\text{mean number of photons}}{(\text{revolution})(\text{particle})} = N = \frac{5\pi}{\sqrt{3}}\,\gamma\,\alpha \quad \Rightarrow \quad \frac{\text{mean energy}}{\text{photon}} = \langle \hbar\,\omega \rangle = \frac{I}{N} = \frac{8}{15\sqrt{3}}\,\hbar\,\omega_c$$

$$\Rightarrow \quad \omega_c \propto \gamma\,, \quad \gamma(\text{GeV}) = O(10^4) \quad \Rightarrow \quad \lambda_{\text{fundamental}} = 2\pi\,\rho \sim \text{hundred of meters}$$

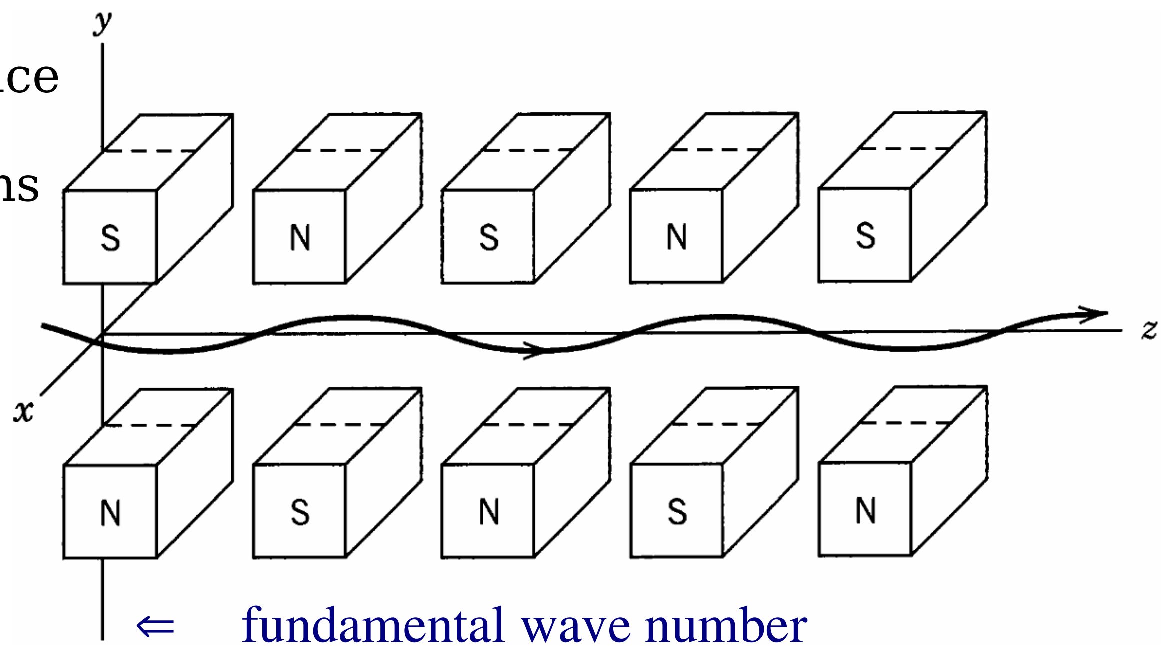
$$\Rightarrow \quad \lambda_{\text{photon}} \sim 10^{-10}\ \text{meter} \Rightarrow \text{keV x-ray}$$

14.7 Undulators and Wigglers for Synchrotron Light Sources

- The magnetic properties of wigglers & undulators make the electrons undergo special motion that results in the concentration of the radiation into a much more monochromatic spectrum or series of separated peaks.

- The essential idea of undulators and wigglers is that a moving relativistically charged particle is caused to move transversely to its general forward motion by magnetic fields that alternate periodically.

- The external magnetic fields induce small transverse oscillations in the motion; the associated accelerations cause radiation to be emitted.



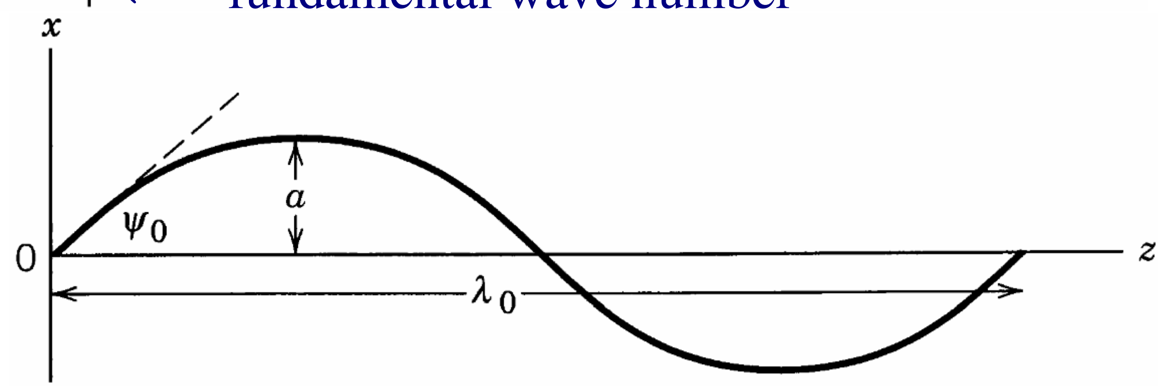
A. Qualitative Features

- $x \approx a\left(B_{\text{wiggler}}, E_{\text{particle}}\right) \sin \dfrac{2\pi z}{\lambda_0}$

$$\Rightarrow \quad \psi_0 = \left.\frac{dx}{dz}\right|_{z=0} = k_0 a \quad \Leftarrow \quad k_0 = \frac{2\pi}{\lambda_0} \quad \Leftarrow \quad \text{fundamental wave number}$$

$$\text{period} \quad T = \frac{\lambda_0}{\beta c} \quad \Rightarrow \quad k_{0,\text{ real}} = \beta k_0$$

$$\text{for} \quad \gamma \gg 1 \quad \Rightarrow \quad k_0 \approx k_{0,\text{ real}}$$



- For $\gamma \gg 1$, the radiation is confined to a width $\Delta\theta = O(1/\gamma)$ about the actual path.

- As the particle moves in its oscillatory path, the "searchlight" beam of radiation will flick back and forth about the forward direction.

(a) Wiggler ($\psi_0 \gg \Delta\theta$)

- $\nu_0 = \frac{\omega_0}{2\pi} = \frac{c\,k_0}{2\pi} = O(10\text{ GHz})$ for $\lambda_0 = O(\text{centimeters})$ The phenomenon is very much as in an ordinary synchrotron with bunches spaced a few centimeters apart.

- The spectrum of radiation extends to frequencies about γ^3 times the basic freq.

$$\text{basic freq.}\quad \Omega = \frac{c}{R} \quad \Leftarrow \quad R: \begin{array}{c}\text{effective radius}\\ \text{of curvature}\end{array} \quad \Rightarrow \quad R_{\min} = \frac{1}{k_0^2\,a} = \frac{\lambda_0}{2\pi\,\psi_0}$$

- The wiggler radiation spectrum is very much like the synchrotron radiation spectrum, with a fundamental frequency Ω, $\Rightarrow \quad \omega_c = \gamma^3\,\Omega \quad \Leftarrow \quad \Omega = 2\pi\,c\,\psi_0/\lambda_0$

- If the wiggler magnet structure has N periods, the intensity of radiation will be N times that for a single pass of a particle in the equivalent circular machine.

- $K \equiv \gamma\,\psi_0 \quad \Rightarrow \quad K \gg 1$ for wiggler $\quad \Rightarrow \quad \omega_c = O\left(\gamma^2 K \frac{2\pi\,c}{\lambda_0}\right) \quad \Leftarrow \quad \lambda_c = O\left(\frac{\lambda_0}{\gamma^2 K}\right)$

(b) Undulators ($\psi_0 \ll \Delta\theta$ or $K \ll 1$)

- If $\psi_0 \ll \Delta\theta$, the searchlight beam of radiation moves negligibly compared to its own angular width.

- the radiation detected by an observer is an almost *coherent superposition* of the contributions from all the oscillations of the trajectory.

- For perfect coherence & an infinite number of magnet periods (and infinitesimal angular resolution of the detector), the radiation would be monochromatic.

- For finite N magnet periods the spread in frequency is $\Delta\omega/\omega = O(1/N)$.

- the frequency spectrum from an undulator is sharply peaked.

- The FitzGerald-Lorentz contraction means that in the particle's rest frame the magnet structure is rushing by the particle with a spatial period λ_0/γ

$$\omega_{\text{rest}} \approx \gamma \frac{2\pi c}{\lambda_0} \quad \Rightarrow \quad \omega_{\text{rest}} = \gamma\,\omega_{\text{lab}}(1-\beta\cos\theta) \approx \omega_{\text{lab}} \frac{1+\gamma^2\theta^2}{2\gamma} \quad \Rightarrow \quad \omega_{\text{lab}} \approx \frac{2\gamma^2}{1+\gamma^2\theta^2}\frac{2\pi c}{\lambda_0}$$

$$\text{For} \quad \gamma\theta \ll 1 \quad \Rightarrow \quad \omega_{\text{lab}} = O(\gamma^2) \quad \text{with fixed} \quad K$$

B. Some Details of the Kinematics and Particle Dynamics

- to consider the particle in its average rest frame, in which it oscillates both transversely and longitudinally.

- Its initial γ and β remain unchanged because **B** does no work on the particle.

- Due to the transverse motion, the particle's average speed in the *z*-direction, $c\,\bar{\beta} < c\,\beta$, and its associated $\bar{\gamma} < \gamma$. The average rest frame moves with speed $c\,\bar{\beta}$.

- length per cycle $s = \int_0^{\lambda_0} \sqrt{1 + (\frac{d x}{d z})^2}\, d z = \int_0^{\lambda_0} [1 + \frac{1}{2}(\frac{d x}{d z})^2 + \cdots]\, d z \approx \lambda_0 (1 + \frac{1}{4}\psi_0^2)$ for $\psi_0 \ll 1$

$$\Rightarrow \quad \bar{\beta} = \frac{\beta}{1 + \psi_0^2/4} \approx \beta (1 - \frac{1}{4}\psi_0^2) \approx 1 \quad \text{for} \quad \beta \approx 1$$

$$\Rightarrow \quad \bar{\gamma}^{-2} = 1 - \bar{\beta}^2 \approx 1 - \beta^2 (1 - \frac{1}{2}\psi_0^2) \approx \gamma^{-2} + \frac{1}{2}\psi_0^2 = \gamma^{-2}(1 + \frac{1}{2}K^2) \quad \Rightarrow \quad \bar{\gamma} = \frac{\gamma}{\sqrt{1 + K^2/2}}$$

Since $K \gg 1$, $\bar{\gamma}$ can differ significantly from γ.

- Lorentz force eqn $\frac{d p_x}{d \tau} = e\gamma [E_x + (\boldsymbol{\beta} \times \mathbf{B})_x] \quad \Rightarrow \quad \ddot{x} = -\frac{e B_y \beta_z}{\gamma m} \quad \Leftarrow \quad \beta, \gamma = \text{const}, \; \beta_y B_z \to 0$

$$z \simeq c t, \; \beta_z \simeq 1 \quad \Rightarrow \quad B_y(z) = -\frac{\gamma m c^2}{e}\frac{d^2 x}{d z^2} = B_0 \sin k_0 z \quad \Leftarrow \quad B_0 = \frac{\gamma m c^2 k_0^2 a}{e}$$

the requisite magnetic structure to have a sinusoidal transverse motion

- $K = \gamma k_0 a = \dfrac{e B_0}{k_0 m c^2} = \dfrac{e B_0 \lambda_0}{2\pi m c^2}$
- An actual magnet structure will be periodic, but not sinusoidal.

- We can make a Fourier decomposition of the actual B_y in multiples of k_0. Each component will contribute to the motion. The fundamental will dominate. For simplicity, we keep only that contribution.

- The longitudinal oscillations can be found from the constancy of β

$$\beta_z(t) \approx \beta - \frac{\beta_x^2}{2\beta} \approx \beta - \frac{\beta_x^2}{2} \quad \Leftarrow \quad \beta_z^2 = \beta^2 - \beta_x^2, \quad |\beta_x| \ll \beta$$

$$\approx \beta - \frac{1}{2} k_0^2 a^2 \cos^2(k_0 c t) \quad \Leftarrow \quad \beta_x \approx k_0 a \cos(k_0 c t) \quad \Leftarrow \quad x = a \sin k_0 z \approx a \sin(k_0 c t)$$

$$= \beta - \frac{1}{4} k_0^2 a^2 [1 + \cos(2\mathrm{k}_0 c t)] = \bar{\beta} - \frac{K^2}{4\gamma^2} \cos(2\mathrm{K}_0 c t)$$

- $$z(t) = \int c\,\beta_z(t)\,\mathrm{d}t = c\bar{\beta}t - \frac{\lambda_0 K^2}{16\pi\gamma^2} \sin(2k_0 c t) \quad \text{longitudinal}$$
  $$x(t) = \int c\,\beta_x(t)\,\mathrm{d}t = \frac{\lambda_0 K}{2\pi\gamma} \sin(k_0 c t) \quad \text{transverse} \qquad (7)$$

C Particle Motion in the Average Rest Frame

- Lorentz transformation

$$x' = x, \quad z' = \bar{\gamma}(z - c\bar{\beta}t), \quad c t' = \bar{\gamma}(c t - \bar{\beta} z) \quad \Rightarrow \quad c t' = \bar{\gamma}\left[c t (1 - \bar{\beta}^2) + \frac{\bar{\beta} K^2}{8 k_0 \gamma^2} \sin 2\theta\right] \quad \Leftarrow \quad (7), \ \theta = k_0 c t$$

$$\Rightarrow \quad t = \bar{\gamma} t' - \frac{1}{4\mathrm{k}_0 c} \frac{K^2}{2 + K^2} \sin(2\bar{\gamma} k_0 c t') \quad \Leftarrow \quad t \approx \bar{\gamma} t' \quad \text{to the 1st approximation}$$

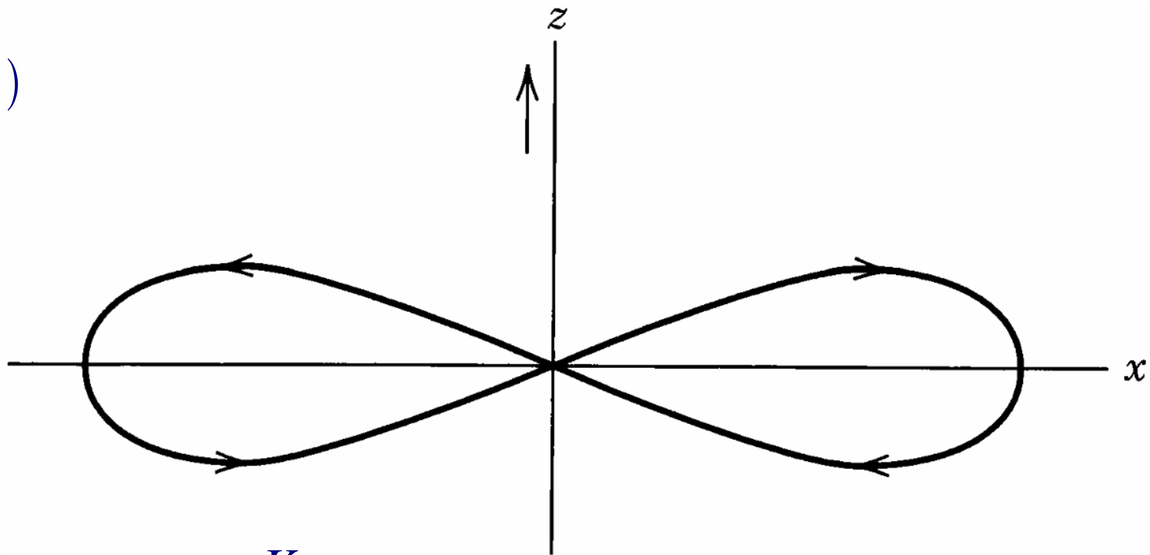
$$\Rightarrow \quad \theta = \bar{\gamma} k_0 c t' - \frac{1}{4} \frac{K^2}{2 + K^2} \sin(2\bar{\gamma} k_0 c t') \quad \Leftarrow$$

usually using the 1st term is good enough
the 2nd term is used in differentiation

$$x'(t') = \frac{K}{\gamma\, k_0} \sin\theta(t') = a \sin\theta(t')$$

$$\Rightarrow \quad z'(t') = -\frac{\bar{\gamma}\, K^2}{8\,\gamma^2\, k_0} \sin 2\theta(t')$$

$$= -\frac{K\, a}{8\sqrt{1+k^2/2}} \sin 2\theta(t')$$



$$\Rightarrow \quad z' = \mp 2\, z'_{\max} \frac{x'}{a} \sqrt{1 - \frac{x'^2}{a^2}} \quad \Leftarrow \quad z'_{\max} = \frac{K\, a}{8\sqrt{1+K^2/2}} \quad \Rightarrow \quad \begin{matrix} K \gg 1 & \Rightarrow & \infty\text{-pattern} \\ K \ll 1 & \Rightarrow & \text{1d SHM in } x \end{matrix}$$

- $\begin{bmatrix}\text{particle's speed in} \\ \text{the moving frame}\end{bmatrix}^2 = \beta'^2 = \frac{1}{c^2}\left[\left(\frac{d\,x'}{d\,t'}\right)^2 + \left(\frac{d\,z'}{d\,t'}\right)^2\right]$

$$\Rightarrow \quad \beta'^2 = \left[\frac{2\,K^2}{2+K^2}\cos^2\theta + \frac{K^4}{4\,(2+K^2)^2}\cos^2 2\theta\right]\left[1 - \frac{K^2}{2\,(2+K^2)}\cos 2\theta\right]^2 \quad \Leftarrow \quad \theta = \bar{\gamma}\, k_0\, c\, t' \text{ now}$$

$$\beta' \approx K\cos\theta \qquad \text{for} \quad K \ll 1 \quad \Rightarrow \quad \text{nonrelativistic SHM} \quad \Rightarrow \quad \text{undulator}$$

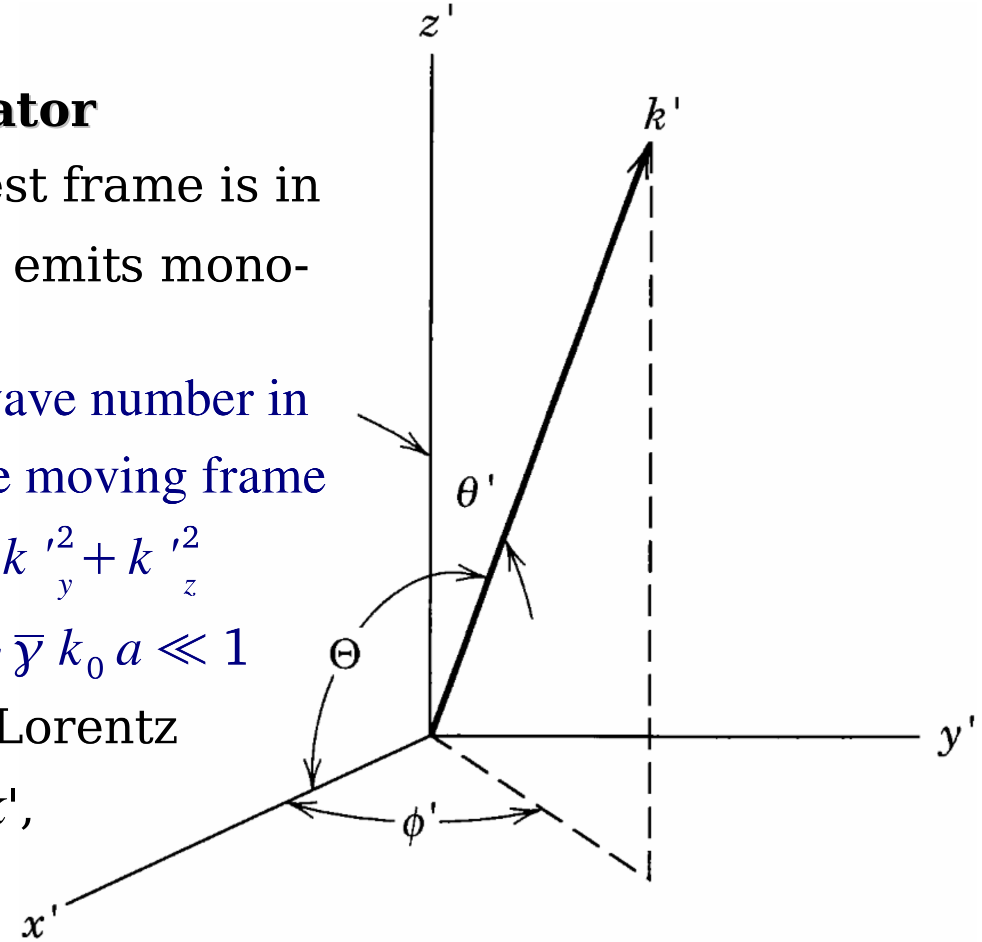
$$\Rightarrow \quad \beta' \approx 1 - \frac{(2\cos^2\theta - 1)^2}{4} \qquad \text{for} \quad K \to \infty \quad \Rightarrow \quad \frac{3}{4} < \beta' < 1 \quad \text{relativistic} \quad \Rightarrow \quad \text{wiggler}$$

- the radiation in the *moving* frame consists of many harmonics of the basic frequency, with an angular distribution that is far from a simple dipole pattern.

- The laboratory radiation pattern from a strong wiggler is better described by the contributions in the direction of observation.

D. Radiation Spectrum from an Undulator

- When $K \ll 1$, the motion in the average rest frame is in nonrelativistic SHM along the x axis and it emits mono-chromatic dipole radiation



$$\frac{dP'}{d\Omega'} = \frac{e^2 c}{8\pi} k'^4 a^2 \sin^2\Theta \quad \Leftarrow \quad k' = \bar{\gamma} k_0$$ wave number in the moving frame

$$= \frac{e^2 c}{8\pi} K^2 (k'^2_y + k'^2_z) \quad \Leftarrow \quad k'^2 \sin^2\Theta = k'^2_y + k'^2_z, \quad K = \gamma k_0 a \approx \bar{\gamma} k_0 a \ll 1$$

- Since the phase-space density d^3k/ω is a Lorentz invariant, it is useful to consider $\omega' d^3P'/d^3k'$,

$$d^3P' \equiv dP' dk' = \frac{dP'}{d\Omega'} \frac{c}{k'} \frac{d^3k'}{\omega'}$$

$$= \left[\frac{e^2 c^2}{8\pi} K^2 (k'^2_y + k'^2_z) \frac{\delta(k' - \bar{\gamma} k_0)}{\bar{\gamma} k_0}\right] \frac{d^3k'}{\omega'} \quad \Leftarrow \quad d^3k' = k'^2 dk' d\Omega'$$

Inserting $\delta(k' - \bar{\gamma} k_0)$ to assure the monochromatic nature

$$\Delta t' = \frac{\lambda_0}{\bar{\gamma}\bar{\beta} c} \approx \frac{\lambda_0}{\bar{\gamma} c}$$ time for passing one period of the magnet structure in the moving frame

$$\Rightarrow \text{No. of photon emitted} = \frac{\Delta t'}{\hbar\omega'} \frac{d^3P'}{d^3k'/\omega'} \Leftarrow N' = N \Rightarrow \frac{\Delta t}{\hbar\omega} \frac{d^3P}{d^3k/\omega} \quad \Leftarrow \quad \text{invariant}$$

$$\Rightarrow \quad \frac{d^3 P}{d^3 k/\omega} = \frac{\omega\, \Delta t'}{\omega'\, \Delta t} \frac{d^3 P'}{d^3 k'/\omega'} = \frac{c\, e^2 K^2}{8\pi \bar{\gamma}^3} \frac{k^2}{k_0^2} \left(k'^2_y + k'^2_z\right) \delta\left(k' - \bar{\gamma}\, k_0\right) \quad \Leftarrow \quad \begin{array}{l} \Delta t / \Delta t' = \bar{\gamma} \\ \dfrac{d^3 k}{\omega} = \dfrac{k\, dk\, d\Omega}{c} \end{array}$$

$$\phi' = \phi$$

$$k'_y = k_y = k \sin\theta \sin\phi$$

in the lab variables $\quad +$

$$k = \frac{k_0}{1 - \bar{\beta}\cos\theta} \quad \Leftarrow \quad k' = \bar{\gamma}\, k_0$$

$$k'_z = \bar{\gamma}\, k\left(\cos\theta - \bar{\beta}\right)$$

$$\bar{\gamma} \gg 1 \quad \Rightarrow \quad \theta \ll 1, \quad \bar{\beta} \approx 1 - \frac{1}{2\,\bar{\gamma}^2}$$

$$k' = \bar{\gamma}\, k\left(1 - \bar{\beta}\cos\theta\right)$$

$$\Rightarrow \quad \frac{d^3 P}{d\eta\, dk\, d\phi} = \frac{c\, e^2 \bar{\gamma}^2 K^2 k_0^2}{2\pi} \frac{(1-\eta)^2 + 4\eta \sin^2\phi}{(1+\eta)^4} \delta\left[k(1+\eta) - 2\bar{\gamma}^2 k_0\right] \quad \Leftarrow \quad \eta = (\bar{\gamma}\,\theta)^2$$

- because of the delta function, the freq and angular distributions are not indep.

(a) Angular Distribution

- $$\frac{d^2 P}{d\eta\, d\phi} = \int \frac{d^3 P}{d\eta\, dk\, d\phi}\, dk = \frac{c\, e^2 \bar{\gamma}^2 K^2 k_0^2}{2\pi} \frac{(1-\eta)^2 + 4\eta \sin^2\phi}{(1+\eta)^5}$$

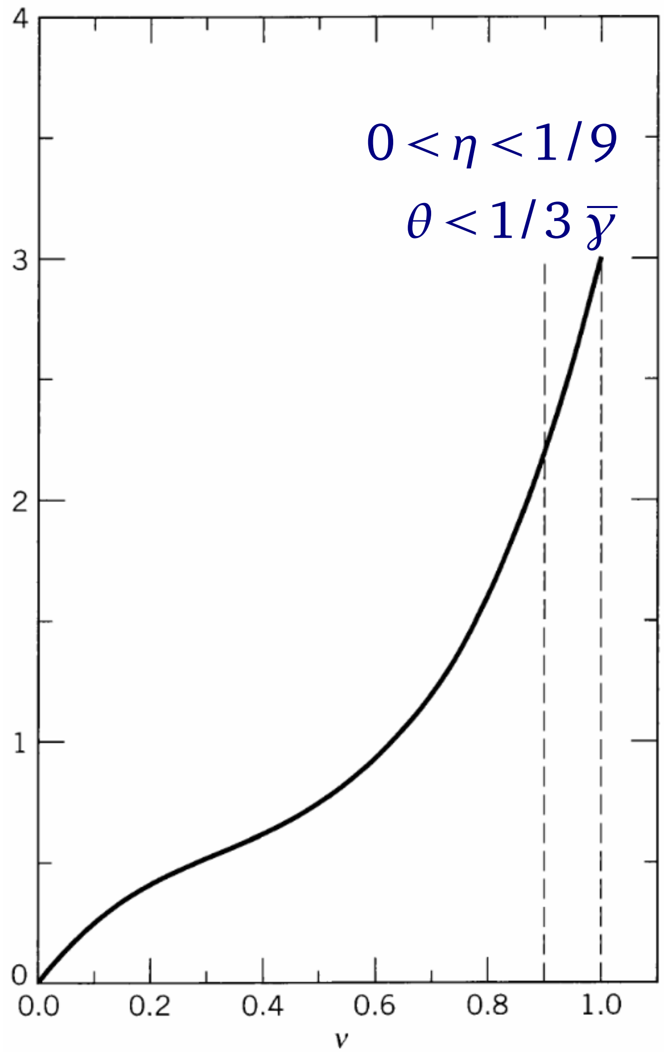
$$\Rightarrow \quad \text{total radiated power} \quad P = \frac{c\, e^2 \bar{\gamma}^2 K^2 k_0^2}{3} \quad \Leftarrow \quad \frac{dP}{d\eta} = \int \frac{d^2 P}{d\eta\, d\phi}\, d\phi = 3P \frac{1+\eta^2}{(1-\eta)^5}$$

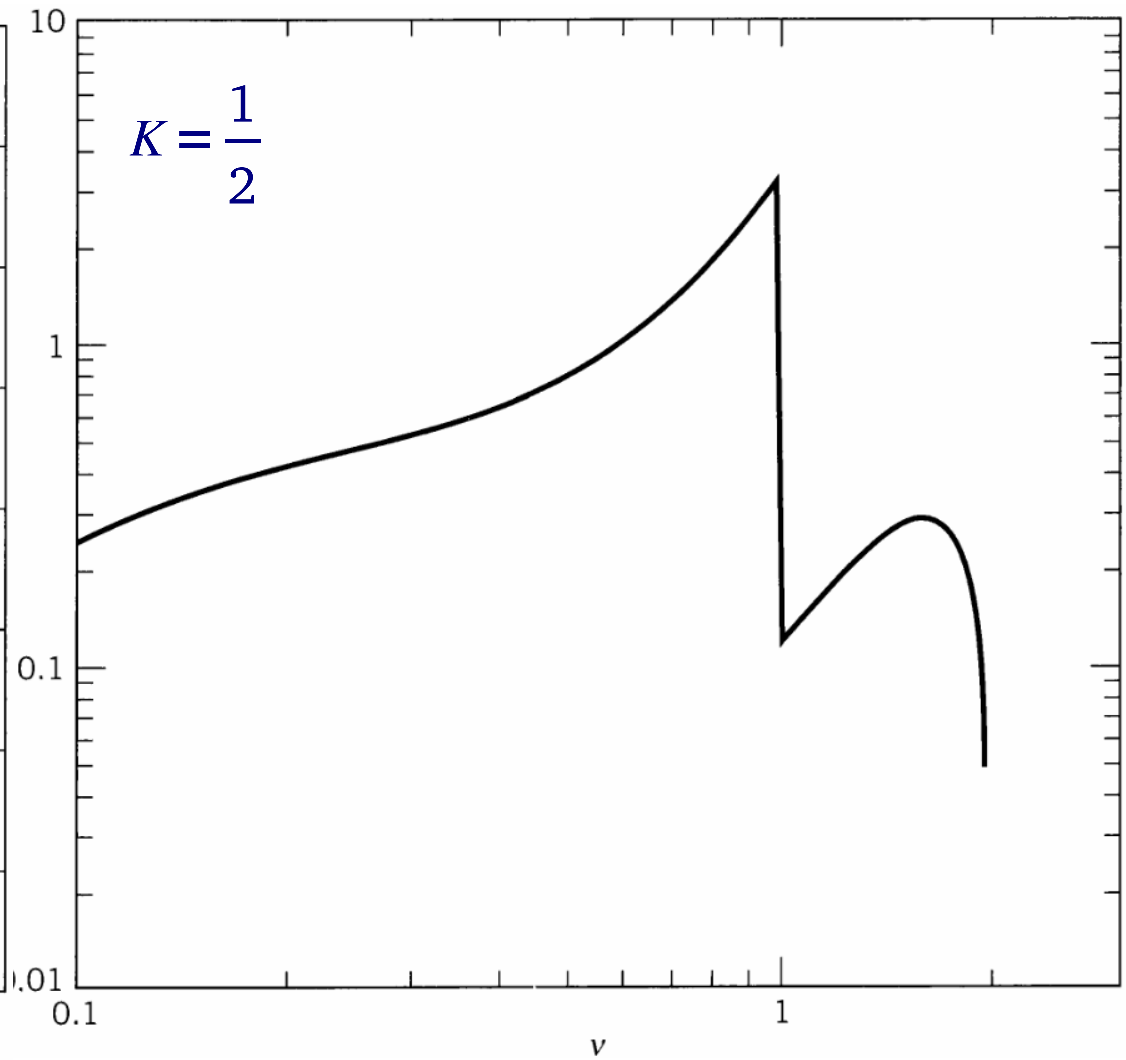
- $\langle \eta \rangle = 1$

(b) Frequency Distribution

- $$\frac{dP}{d\nu} = \int_{\eta_1}^{\eta_2} \frac{d^3 P}{d\eta\, d\nu\, d\phi} d\eta\, d\phi = 3P\nu(1-2\nu+2\nu^2) \;\Leftarrow\; \frac{1}{1+\eta_2} < \nu \left[\equiv \frac{k}{2\bar{\gamma}^2 k_0}\right] < \frac{1}{1+\eta_1}$$
- this spectrum is for perfectly sinusoidal motion of the particle at all times.

- If N of magnet periods is finite, the duration of the oscillatory motion is finite; the wave train will have a fractional spread in frequency of the order of 1/N.





- For large N the spread is small compared to the spread from finite acceptance.

- For small K, there are higher harmonics, coming from higher multipoles caused by the ∞-pattern motion.

- The 2nd harmonic comes from a coherent superposition of the fields of a dipole in the z-direction [z'$\propto$sin2θ(t')] and a quadrupole caused by the x' motion.

(c) Energy of Photons and Number Emitted per Magnet Period

- $\hbar\,\omega_{\text{max}} = 2\,\hbar\,\bar{\gamma}^2\,k_0\,c \quad \text{at} \quad \eta = 0 \quad + \quad$ energy radiated per magnet period $\Delta E = P\,\Delta t \quad \Leftarrow \quad \Delta t = \dfrac{\lambda_0}{c}$

$$\Rightarrow \quad \text{No. of photon per magnet period} \quad N_\gamma \geq \frac{P\,\Delta t}{\hbar\,\omega_{\text{max}}} = O\left(\alpha\,K^2\right) \quad \Rightarrow \quad N_\gamma = \frac{2\,\pi}{3}\,\alpha\,K^2 \quad \Leftarrow \quad (8)$$

E. Numerical Values and Representative Spectra and Facilities

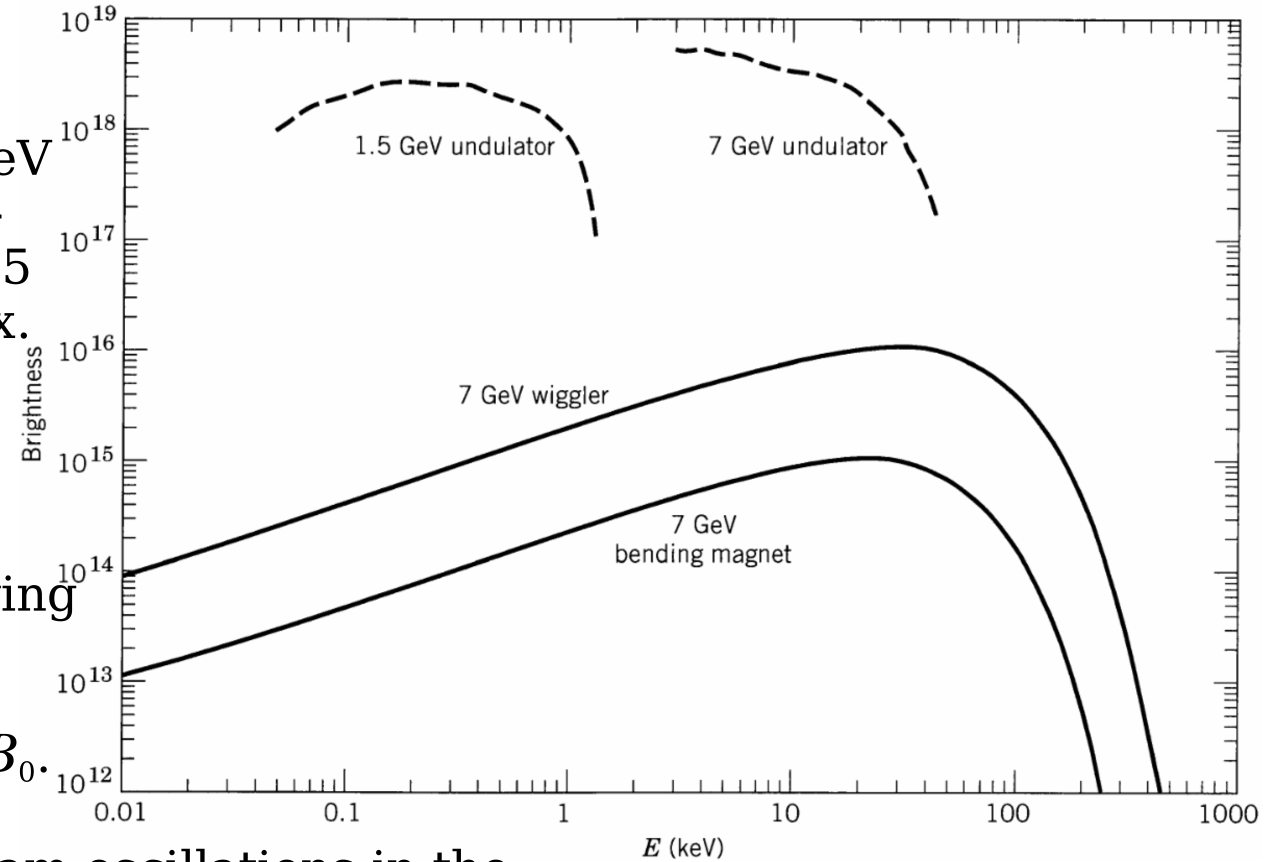
- The parameters K and $\hbar\omega_{\text{max}}$ are given for electrons

$$K = \frac{e\,B_0}{k_0\,m\,c^2} = \frac{e\,B_0\,\lambda_0}{2\,\pi\,m\,c^2}\,93.4\,B_0\,(T)\,\lambda_0\,(\text{m}), \quad \hbar\,\omega_{\text{max}}\,(\text{eV}) = \frac{9.496\,[E\,(\text{GeV})]^2}{(1 + K^2/2)\,\lambda_0\,(\text{m})}$$

$$\Rightarrow \quad \text{Typical undulator: } B_0 \sim 0.5\,T\,, \quad \lambda_0 \sim 4\text{ cm}\,, \quad E \sim 1 - 7\text{ GeV} \quad \Rightarrow \quad K \sim 2, \quad \hbar\,\omega_{\text{max}} \sim 80\text{ eV- }4\text{ keV}$$

$$\text{Typical wiggler: } B_0 \sim 1\,T\,, \quad \lambda_0 \sim 20\text{ cm} \quad \Rightarrow \quad K \sim 20$$

- The lower energy facilities provide photons in the tens of eV to several KeV range; the high-energy facilities extend to 10-75 keV, and higher at reduced flux.



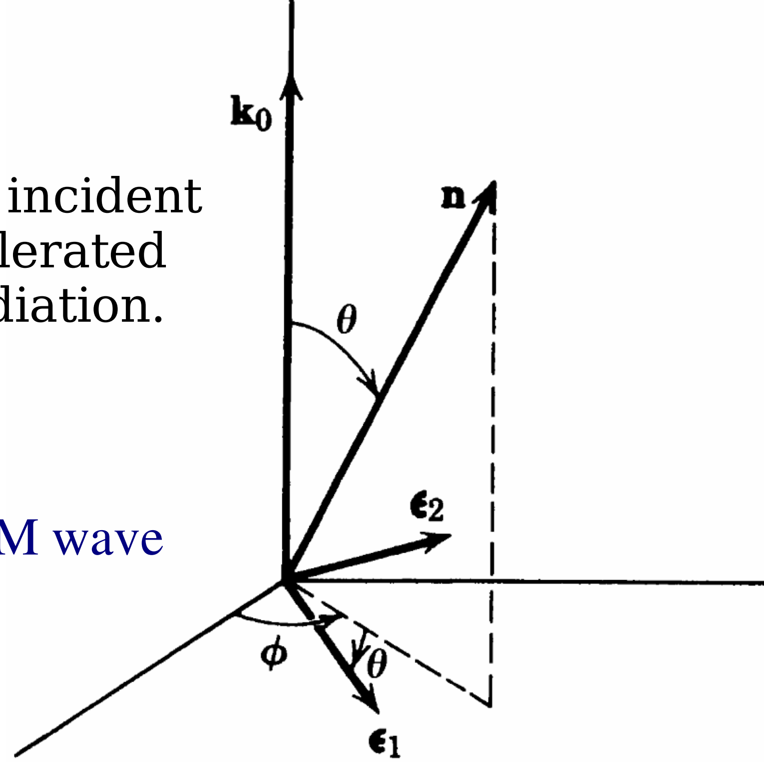
F. Additional Comments

- An undulator's fundamental freq. ω_{max} can be tuned by varying K by changing the gap in the magnet structure & changing B_0.

- The simple undulator with beam oscillations in the horizontal plane provides linearly polarized light. Circular polarization can be provided by use of a designed helical undulator. Or, 2 undulators at right angles with an adjustable longitudinal spacing between them can be used to produce circular polarization or any other state.

- Free electron lasers are related to wigglers and undulators. An undulator can be thought of as radiating in the forward direction at freq. ω_{max} by spontaneous emission. Addition of a co-traveling EM wave of the same frequency provides the possibility of interaction and stimulated emission and growth of the wave.

14.8 Thomson Scattering of Radiation



- If a plane wave of monochromatic EM radiation is incident On a free charged particle, the particle will be accelerated and so emit radiation—scattering of the incident radiation.

- $\mathbf{E}(\mathbf{x}, t) = \boldsymbol{\epsilon}_0 E_0 e^{i(\mathbf{k}_0 \cdot \mathbf{x} - \omega t)}$

$$\Rightarrow \quad \dot{\mathbf{v}}(t) = \boldsymbol{\epsilon}_0 \frac{e}{m} E_0 e^{i(\mathbf{k}_0 \cdot \mathbf{x} - \omega t)} \quad \Leftarrow \quad \boldsymbol{\epsilon}_0 : \text{polarization of EM wave}$$

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3} |\boldsymbol{\epsilon}^* \cdot \dot{\mathbf{v}}|^2 \quad \Leftarrow \quad \boldsymbol{\epsilon} : \text{polarization of radiation}$$

$$\Rightarrow \quad \left\langle \frac{dP}{d\Omega} \right\rangle = \frac{c}{8\pi} \frac{e^4}{m^2 c^4} |E_0|^2 |\boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_0|^2 \quad \Leftarrow \quad \langle |\dot{\mathbf{v}}|^2 \rangle = \frac{1}{2} \Re (\dot{\mathbf{v}} \cdot \dot{\mathbf{v}}^*)$$

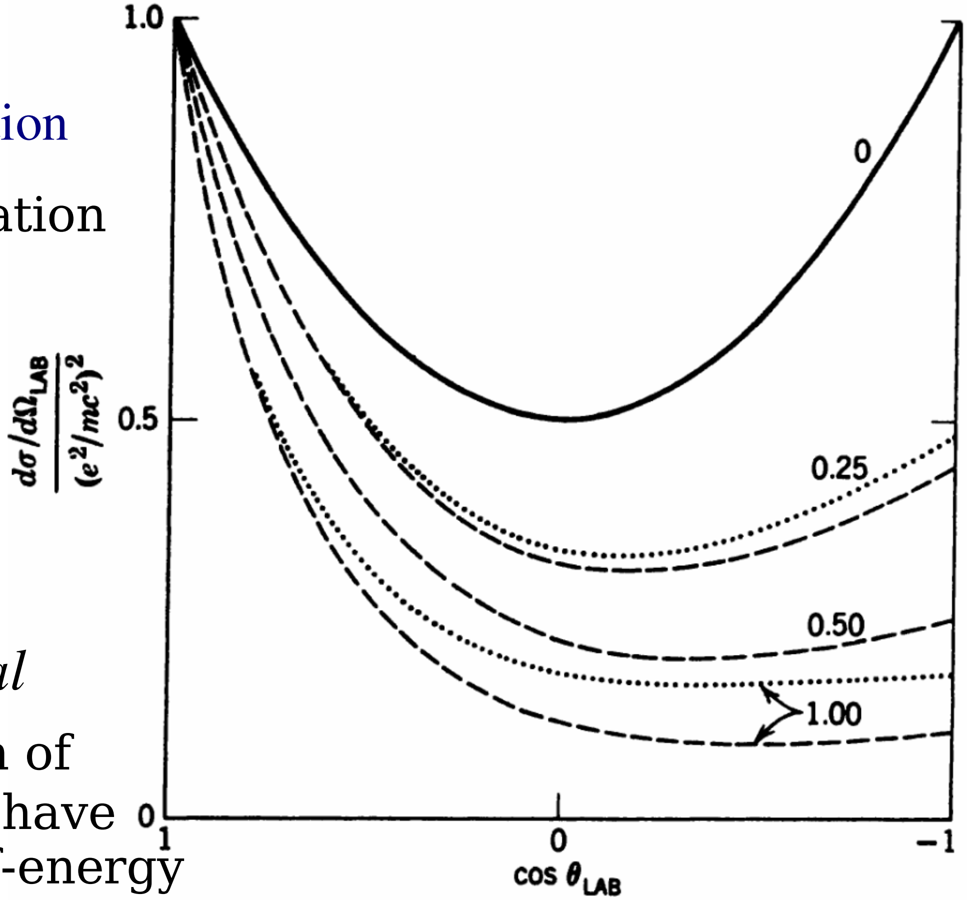
$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{\text{Energy radiated/time/solid angle}}{\text{Incident energy flux in energy/area/time}} = \frac{dP/d\Omega}{c|E_0|^2/8\pi} = \frac{e^4}{m^2 c^4} |\boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_0|^2$$

$$\boldsymbol{\epsilon}_1 = \cos\theta (\mathbf{e}_x \cos\phi + \mathbf{e}_y \sin\phi) - \mathbf{e}_z \sin\theta, \quad \boldsymbol{\epsilon}_2 = -\mathbf{e}_x \sin\phi + \mathbf{e}_y \cos\phi$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{e^4}{m^2 c^4} \cdot \begin{cases} (\cos^2\theta \cos^2\phi + \sin^2\phi) & \text{linear polarization} \parallel x\text{-axis} \\ (\cos^2\theta \sin^2\phi + \cos^2\phi) & \text{linear polarization} \parallel y\text{-axis} \end{cases}$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{e^4}{m^2 c^4} \frac{1 + \cos^2\theta}{2} \quad \text{unpolarized} \quad \Leftarrow \quad \text{Thomson formula}$$

$$\Rightarrow \quad \sigma_T = \frac{8\pi}{3}\frac{e^4}{m^2c^4} \quad \Leftarrow \quad \text{Thomson cross section}$$



- Thomson formula is for scattering of radiation by a free charge, and is appropriate for the scattering of x-rays by electrons or γ-rays by protons.

- The Thomson cross section is equal to 0.665×10^{-24} cm^2 for electrons.

- e^2/mc^2=2.82×10^{-13}cm is called the *classical electron radius* since a classical distribution of charge totaling the electronic charge must have a radius of this order if its electrostatic self-energy is to equal the electron mass.

- The classical Thomson formula is valid only at low frequencies where the momentum of the incident photon can be ignored.

- When the photon's momentum $\hbar\omega/c$ becomes comparable to or larger than mc, modifications occur—quantum-mechanical effects.

- The energy or momentum of the scattered photon is less than the incident energy because the charged particle recoils during the collision.

- $$\frac{k'}{k} = \frac{mc^2}{mc^2 + \hbar\omega(1-\cos\theta)}$$ Compton formula $\Leftarrow$ θ : scattering angle in the lab

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{e^4}{m^2c^4}\frac{k'^2}{k^2}\left|\boldsymbol{\epsilon}^*\cdot\boldsymbol{\epsilon}_0\right|^2$$ $\Leftarrow$ spinless particle

- $(k'/k)^2$ comes entirely from the phase space. Its presence causes the differential cross section to decrease relative to the Thomson result at large angles.

$$\Rightarrow \quad \frac{\sigma}{\sigma_T} = \begin{cases} 1 - 2\dfrac{\hbar\omega}{mc^2} + \cdots & \text{for} \quad \hbar\omega \ll mc^2 \\ \left[\begin{array}{ll} \dfrac{3}{4}\dfrac{mc^2}{\hbar\omega} & \text{spinless} \\ \dfrac{3}{4}\dfrac{mc^2}{\hbar\omega}\left(\dfrac{1}{4} + \dfrac{1}{2}\ln\dfrac{2\hbar\omega}{mc^2}\right) & \text{electron} \end{array}\right. & \text{for} \quad \hbar\omega \gg mc^2 \end{cases}$$

- For protons the departures from the Thomson formula occur at $\hbar\omega > 100$ MeV. This is far below the critical energy $\hbar\omega \sim Mc^2 \sim 1$GeV.

- The reason is that a proton is not a point particle but having a spread-out charge distribution by the strong interactions.