Chapter 8 Waveguides, Resonant Cavities, Optical Fibers

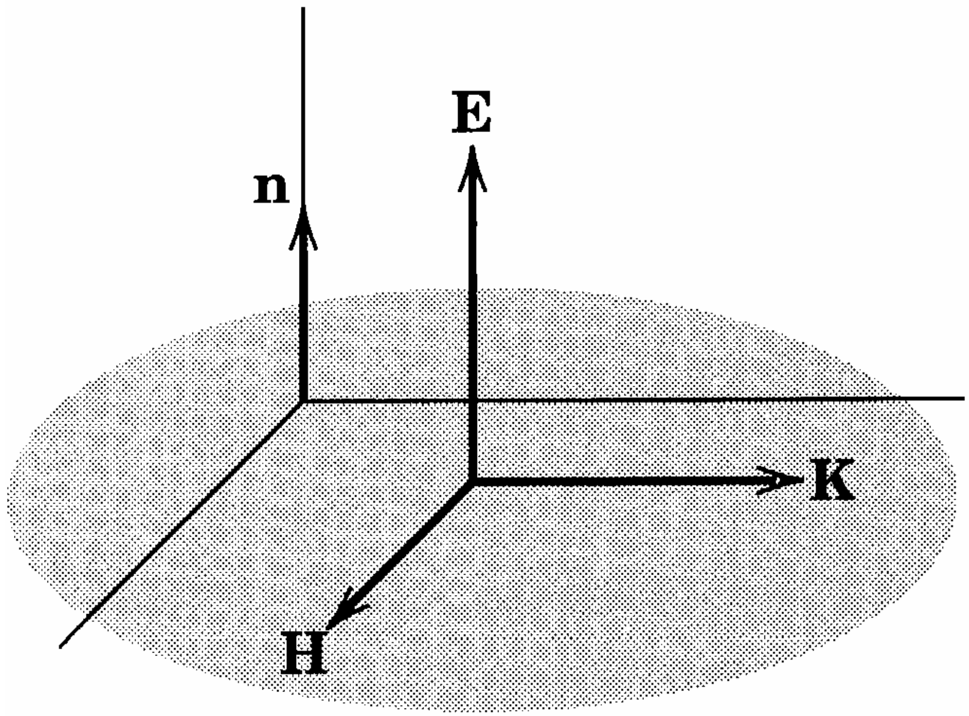
- At high frequencies with wavelengths~meters or less, generating & transmitting EM radiation involves metallic structures with dimensions~the wavelengths.

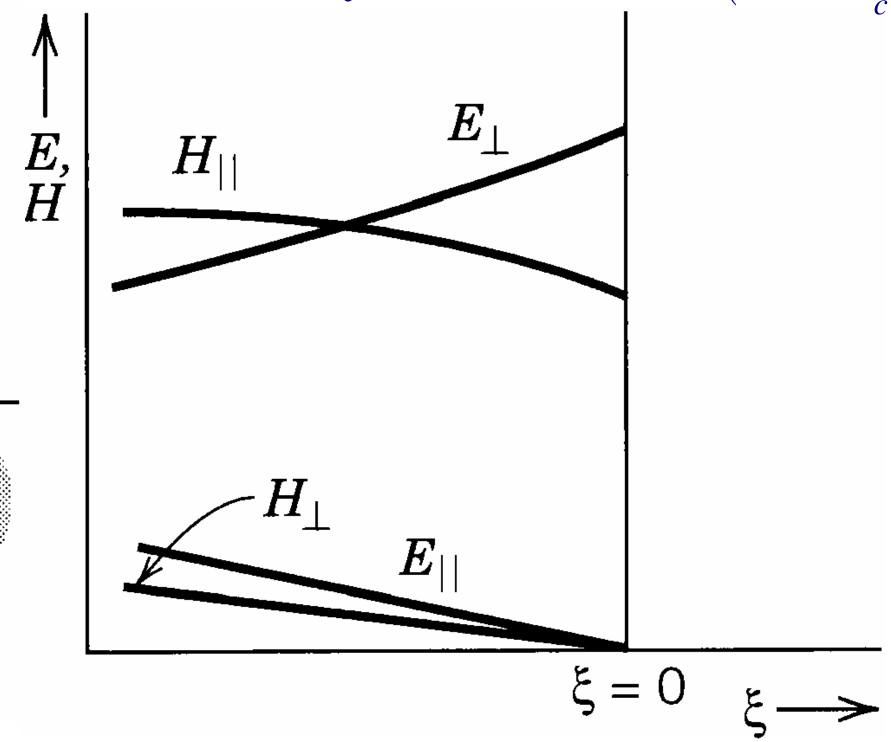
- At much higher (infrared) frequencies, dielectric optical fibers are exploited in the telecommunications industry.

8.1 Fields at the Surface of and Within a Conductor

- Consider a surface with unit normal **n** outward from a perfect conductor into a nonconducting medium

$$\left.\begin{aligned}\mathbf{E}&=0\\ \mathbf{B}&=0\end{aligned}\right\}\text{ in the coductor } \Rightarrow \begin{aligned}\mathbf{n}\cdot\mathbf{D}&=\Sigma \quad \text{surface charge density}\\ \mathbf{n}\times\mathbf{H}&=\mathbf{K} \quad \text{surface current density}\end{aligned} \quad (1) \quad + \quad \begin{aligned}\mathbf{n}\cdot(\mathbf{B}-\mathbf{B}_c)&=0\\ \mathbf{n}\times(\mathbf{E}-\mathbf{E}_c)&=0\end{aligned}$$





- outside the surface of a perfect conductor only *normal* **E** and *tangential* **H** fields can exist, and that the fields drop abruptly to zero inside the perfect conductor.

- inside a good (not perfect) conductor the fields are attenuated exponentially in the skin depth δ. For moderate frequencies, δ <1cm. So the boundary condition (1) are approximately true, aside from a thin transitional layer at the surface.

- By $\boldsymbol{J}=\sigma\mathbf{E}$, with a finite conductivity there can't be a surface layer of current but

$$\mathbf{n}\times(\mathbf{H}-\mathbf{H}_c)=0 \quad \Rightarrow \quad \mathbf{H}_{\parallel}=\mathbf{H}_{\parallel c}$$

- First assume that just outside the conductor there exists only $\mathbf{E}_{\perp}$ & $\mathbf{H}_{\parallel}$, as for a perfect conductor.

- Then use the boundary conditions & Maxwell's eqns in the conductor to find the fields within the transition layer and small corrections to the fields outside.

- the spatial variation of the fields $\perp$ to the surface is much more rapid than the variations $\parallel$ to the surface in solving the Maxwell eqns within the conductor $\Rightarrow$ neglect all derivatives with respect to coordinates $\parallel$ to the surface.

- $$\begin{aligned} &\nabla\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}=\boldsymbol{J} & & \mathbf{E}_c\simeq\frac{1}{\sigma}\nabla\times\mathbf{H}_c \quad \Leftarrow \quad \text{neglect the displacement current} \\ & & \Rightarrow & \\ &\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0 & & \mathbf{H}_c\simeq-\frac{i}{\mu_c\,\omega}\nabla\times\mathbf{E}_c \end{aligned}$$

$$\nabla \simeq -\mathbf{n}\frac{\partial}{\partial \xi} \Leftarrow \begin{array}{l}\mathbf{n}: \text{unit normal outward}\\ \xi: \text{normal coordinate inward}\end{array} \Rightarrow \begin{array}{l}\mathbf{E}_c \simeq -\dfrac{1}{\sigma}\,\mathbf{n}\times\dfrac{\partial \mathbf{H}_c}{\partial \xi}\\ \mathbf{H}_c \simeq \dfrac{i}{\mu_c\,\omega}\,\mathbf{n}\times\dfrac{\partial \mathbf{E}_c}{\partial \xi}\end{array}$$

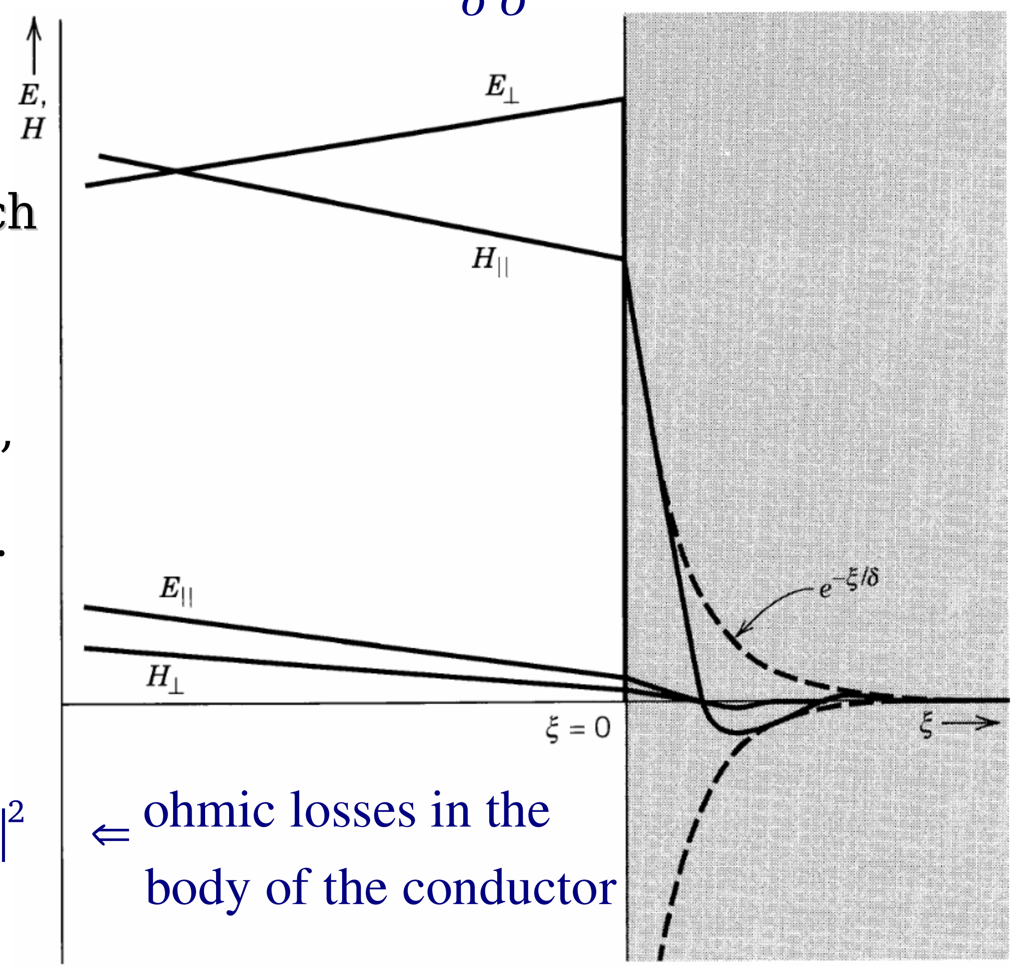
$$\Rightarrow \quad \begin{array}{l}\dfrac{\partial^2}{\partial \xi^2}(\mathbf{n}\times\mathbf{H}_c)+\dfrac{2\,i}{\delta^2}\,\mathbf{n}\times\mathbf{H}_c \simeq 0\\ \mathbf{n}\cdot\mathbf{H}_c \simeq 0\end{array} \Leftarrow \quad \delta = \sqrt{\frac{2}{\mu_c\,\omega\,\sigma}} \quad \Rightarrow \begin{array}{l}\mathbf{H}_c = \mathbf{H}_{\parallel}\, e^{\xi(i-1)/\delta}\\ \mathbf{E}_c \simeq \dfrac{1-i}{\delta\,\sigma}\,\mathbf{n}\times\mathbf{H}_{\parallel}\, e^{\xi(i-1)/\delta}\end{array}$$

- **H** & **E** inside the conductor exhibit the properties of rapid exponential decay, phase difference, and magnetic field much larger than the electric field.

- For a good conductor, the fields inside are $\parallel$ to the surface and propagate $\perp$ to it, with magnitude that depend only on the tangential magnetic field $\mathbf{H}_{\parallel}$ just outside.

- $$\mathbf{E}_{\parallel} = \mathbf{E}_c(\xi = 0) = \frac{1-i}{\delta\,\sigma}\,\mathbf{n}\times\mathbf{H}_{\parallel} \qquad (13)$$

$$\Rightarrow \quad \frac{\mathrm{d}\,P_{\text{loss}}}{\mathrm{d}\,a} = -\frac{1}{2}\,\Re\left[\mathbf{n}\cdot\mathbf{E}\times\mathbf{H}^*\right] = \frac{\mu_c\,\omega\,\delta}{4}\,|\mathbf{H}_{\parallel}|^2 \quad \Leftarrow \text{ohmic losses in the body of the conductor}$$



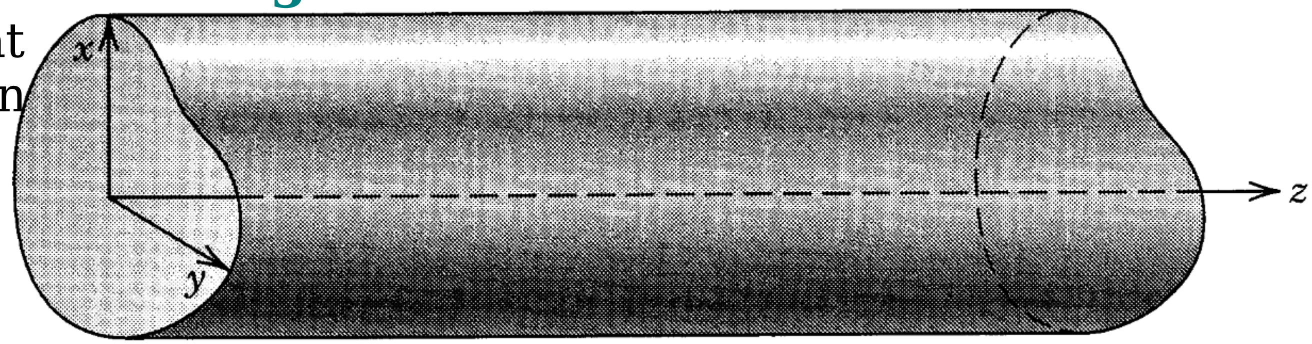
- $\boldsymbol{J} = \sigma\, \mathbf{E}_c = \dfrac{1-i}{\delta}\, \mathbf{n} \times \mathbf{H}_{\parallel}\, e^{\xi (i-1)/\delta}$

$$\Rightarrow \quad \frac{\mathrm{d}\, P_{\text{loss}}}{\mathrm{d}\, a} = \int \frac{\mathrm{d}\, P_{\text{loss}}}{\mathrm{d}\, v}\, \mathrm{d}\,\xi = \int \frac{1}{2}\, \boldsymbol{J} \cdot \mathbf{E}^{*}\, \mathrm{d}\,\xi = \int \frac{1}{2\,\sigma} \left|\boldsymbol{J}\right|^2 \mathrm{d}\,\xi = \frac{\mu_c\, \omega\, \delta}{4} \left|\mathbf{H}_{\parallel}\right|^2$$

- $\mathbf{K}_{\text{eff}} = \int_0^{\infty} \boldsymbol{J}\, \mathrm{d}\,\xi = \mathbf{n} \times \mathbf{H}_{\parallel} \quad \Rightarrow \quad \dfrac{\mathrm{d}\, P_{\text{loss}}}{\mathrm{d}\, a} = \dfrac{1}{2\,\sigma\,\delta} \left|\mathbf{K}_{\text{eff}}\right|^2 \quad \Leftarrow \quad \dfrac{1}{\sigma\,\delta}$: surface resistance of the conductor

- a good conductor behaves effectively like a perfect conductor, with the idealized surface current replaced by an effective surface current, which is distributed throughout a very small, but finite, thickness at the surface.

8.2 Cylindrical Cavities and Waveguides



- A practical situation of great importance is the propagation or excitation of EM waves in hollow metallic cylinders.

- If the cylinder has end surfaces,it is called a cavity; otherwise, a waveguide.

- The boundary surfaces are assumed to be perfect conductors.

- $$\begin{array}{ll} \nabla\times\mathbf{E}=i\,\omega\,\mathbf{B}\,, & \nabla\cdot\mathbf{B}=0 \\ \nabla\times\mathbf{B}=-i\,\mu\,\epsilon\,\omega\,\mathbf{E}\,, & \nabla\cdot\mathbf{E}=0 \end{array} \Leftarrow \begin{array}{l}\text{Maxwell's eqns}\\ \text{inside the cylinder}\end{array} \Rightarrow (\nabla^2+\mu\,\epsilon\,\omega^2)\begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix}=0$$

$$\Rightarrow \begin{array}{l} \mathbf{E}(x,y,z,t)=\mathbf{E}(x,y)\,e^{i(\pm k z-\omega t)} \\ \mathbf{B}(x,y,z,t)=\mathbf{B}(x,y)\,e^{i(\pm k z-\omega t)} \end{array} \Rightarrow (\nabla_t^2+\mu\,\epsilon\,\omega^2-k^2)\begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix}=0 \Leftarrow \begin{array}{l}\nabla_t^2=\nabla^2-\partial_z^2\\ \text{transverse part}\end{array}$$

$$\Rightarrow \begin{array}{l} \mathbf{E}=\mathbf{E}_z+\mathbf{E}_t \\ \mathbf{B}=\mathbf{B}_z+\mathbf{B}_t \end{array} \Leftarrow \begin{array}{ll} \mathbf{E}_z=E_z\,\hat{\mathbf{z}}\,, & \mathbf{E}_t=(\hat{\mathbf{z}}\times\mathbf{E})\times\hat{\mathbf{z}} \\ \mathbf{B}_z=B_z\,\hat{\mathbf{z}}\,, & \mathbf{B}_t=(\hat{\mathbf{z}}\times\mathbf{B})\times\hat{\mathbf{z}} \end{array}$$

$$\Rightarrow \begin{array}{lll} \dfrac{\partial\,\mathbf{E}_t}{\partial z}+i\,\omega\;\hat{\mathbf{z}}\times\mathbf{B}_t=\nabla_t E_z\,, & \hat{\mathbf{z}}\cdot(\nabla_t\times\mathbf{E}_t)=i\;\omega\,B_z\,, & \nabla_t\cdot\mathbf{E}_t=-\dfrac{\partial\,E_z}{\partial z} \\ \dfrac{\partial\,\mathbf{B}_t}{\partial z}-i\,\mu\,\epsilon\,\omega\,\hat{\mathbf{z}}\times\mathbf{E}_t=\nabla_t B_z\,, & \hat{\mathbf{z}}\cdot(\nabla_t\times\mathbf{B}_t)=i\,\mu\,\epsilon\,\omega\,E_z\,, & \nabla_t\cdot\mathbf{B}_t=-\dfrac{\partial\,B_z}{\partial z} \end{array} \qquad (6)$$

- If E_z and B_z are known the transverse components of **E** and **B** are determined

$$\mathbf{E}_t = \frac{i}{\mu\epsilon\omega^2 - k^2}\left[\pm k\nabla_t E_z - \omega\hat{\mathbf{z}}\times\nabla_t B_z\right]$$

$$(2) \quad \Leftarrow \quad \text{for} \quad E_z \neq 0 \quad \text{and/or} \quad B_z \neq 0$$

$$\mathbf{B}_t = \frac{i}{\mu\epsilon\omega^2 - k^2}\left[\pm k\nabla_t B_z + \mu\epsilon\omega\hat{\mathbf{z}}\times\nabla_t E_z\right]$$

- the *transverse electromagnetic* (TEM) wave: only field components transverse to the direction of propagation $\Leftarrow$ a degenerate solution;

$$E_z = 0 \quad \Rightarrow \quad \mathbf{E}_{\mathrm{TEM}} = \mathbf{E}_t \quad \Rightarrow \quad \nabla_t\times\mathbf{E}_{\mathrm{TEM}} = 0, \quad \nabla_t\cdot\mathbf{E}_{\mathrm{TEM}} = 0$$

$$B_z = 0 \qquad \mathbf{B}_{\mathrm{TEM}} = \mathbf{B}_t \qquad \nabla_t\times\mathbf{B}_{\mathrm{TEM}} = 0, \quad \nabla_t\cdot\mathbf{B}_{\mathrm{TEM}} = 0$$

- $\mathbf{E}_{\mathrm{TEM}}$ is a solution of an *electrostatic* problem in 2d: $\Leftarrow \quad \nabla_t^2\mathbf{E}_{\mathrm{TEM}} = 0$

$$(1) \quad k = k_0 = \omega\sqrt{\mu\epsilon} \quad \Leftarrow \quad (\nabla_t^2 + \mu\epsilon\omega^2 - k^2)\,\mathbf{E}_{\mathrm{TEM}} = 0$$

$$\Rightarrow \quad (2) \quad \mathbf{B}_{\mathrm{TEM}} = \pm\sqrt{\mu\epsilon}\,\hat{\mathbf{z}}\times\mathbf{E}_{\mathrm{TEM}} \quad \text{for waves propagating as} \quad e^{\pm ikz - i\omega t}$$

(3) TEM mode can't exist inside a single, hollow, cylindrical perfect conductor

$$\Leftarrow \quad \mathbf{E}_{\mathrm{inside}} = 0 \quad \Leftarrow \quad \text{The surface is an equipotential}$$

- It is necessary to have 2 or more cylindrical surfaces to support the TEM mode, e.g., The coaxial cable and the parallel-wire transmission line.

- the TEM mode is the absence of a cutoff frequency. The wave number is real for all ω. This is not true for the other modes.

- For a perfectly conducting cylinder ($\mathbf{E}_c=\mathbf{B}_c=0$) the boundary conditions are

$$\begin{matrix} \mathbf{n}\times\mathbf{E}=0 \\ \mathbf{n}\cdot\mathbf{B}=0 \end{matrix} \quad\Rightarrow\quad \begin{matrix} E_{z|S}=E_{\ell|S}=0 \\ B_{n|S}=0 \end{matrix} \quad\Rightarrow\quad \left.\frac{\partial B_z}{\partial n}\right|_S=0 \quad\Leftarrow\quad \mathbf{n}\cdot\left(\frac{\partial \mathbf{B}_t}{\partial z}-i\mu\epsilon\omega\,\hat{\mathbf{z}}\times\mathbf{E}_t=\nabla_t B_z\right)$$

- The 2d wave equations for E_z and B_z, together with the boundary conditions, specify eigenvalue problems of the usual sort.

- For a given frequency ω, only certain values of wave number k can occur (waveguide). For a given k, only certain ω values are allowed (resonant cavity).

- *Since the boundary conditions on E_z and B_z are different, the eigenvalues will in general be different.*

- Transverse Magnetic (TM) Waves: $B_z=0$ everywhere; boundary condition $E_{z|S}=0$

  Transverse Electric (TE) Waves: $E_z=0$ everywhere; boundary condition $\left.\frac{\partial B_z}{\partial n}\right|_S=0$

- The various TM and TE waves, plus the TEM wave if it can exist, constitute a complete set to describe an arbitrary EM disturbance in a waveguide or cavity.

8.3 **Waveguides**

- For the propagation of waves inside a hollow waveguide of uniform cross section, it is found from (2) that

$$\mathbf{H}_t = \pm\frac{\hat{\mathbf{z}}\times\mathbf{E}_t}{Z} \quad (3) \quad \Leftarrow \quad Z = \begin{cases} \dfrac{k}{\epsilon\,\omega} = \dfrac{k}{k_0}\sqrt{\dfrac{\mu}{\epsilon}} & (\mathrm{TM}) \\[2ex] \dfrac{\mu\,\omega}{k} = \dfrac{k_0}{k}\sqrt{\dfrac{\mu}{\epsilon}} & (\mathrm{TE}) \end{cases} \quad \text{wave impedance} \quad \Leftarrow \quad k_0 = \omega\sqrt{\mu\,\epsilon}$$

$$\Rightarrow \begin{matrix} \text{TM Waves} \quad \mathbf{E}_t = \pm\dfrac{i\,k}{\gamma^2}\nabla_t\,\psi \quad \Leftarrow \quad E_z = \psi\, e^{\pm i\,k\,z} \\[2ex] \text{TE Waves} \quad \mathbf{H}_t = \pm\dfrac{i\,k}{\gamma^2}\nabla_t\,\psi \quad \Leftarrow \quad H_z = \psi\, e^{\pm i\,k\,z} \end{matrix} \quad (4) \quad \Leftarrow \quad \begin{matrix} (\nabla_t^2 + \gamma^2)\,\psi = 0, \quad \gamma^2 = \mu\,\epsilon\,\omega^2 - k^2 \\[2ex] \psi_{|S} = 0 \quad (\mathrm{TM}), \quad \dfrac{\partial\,\psi}{\partial\,n}\Big|_S = 0 \quad (\mathrm{TE}) \end{matrix}$$

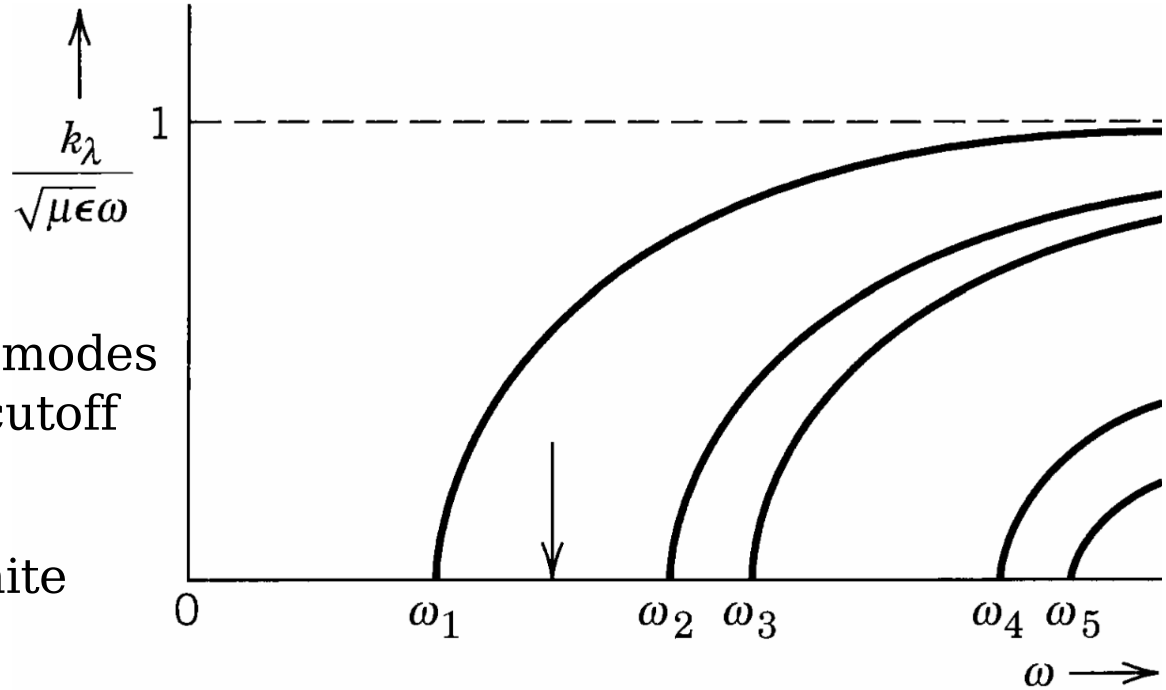
- The elliptic eqn of ψ, with boundary conditions, specifies an eigenvalue problem.

- γ^2 must be nonnegative because ψ must be oscillatory to satisfy the boundary condition on opposite sides of the cylinder.

- There will be a spectrum of eigenvalues γ_λ^2 and corresponding solutions ψ_λ which form an orthogonal set — *modes of the guide*.

$$\text{For a given } \omega, \quad k_\lambda^2 = \mu\,\epsilon\,\omega^2 - \gamma_\lambda^2 \quad \Rightarrow \quad \textit{cutoff freq.} \quad \omega_\lambda = \frac{\gamma_\lambda}{\sqrt{\mu\,\epsilon}} \quad \Rightarrow \quad k_\lambda = \sqrt{\mu\,\epsilon}\,\sqrt{\omega^2 - \omega_\lambda^2}$$

- For $\omega>\omega_\lambda$, then k_λ is real; waves of the λ mode can propagate in the guide.



- For $\omega<\omega_\lambda$, k_λ is imaginary; such modes cannot propagate and are called cutoff modes or evanescent modes.

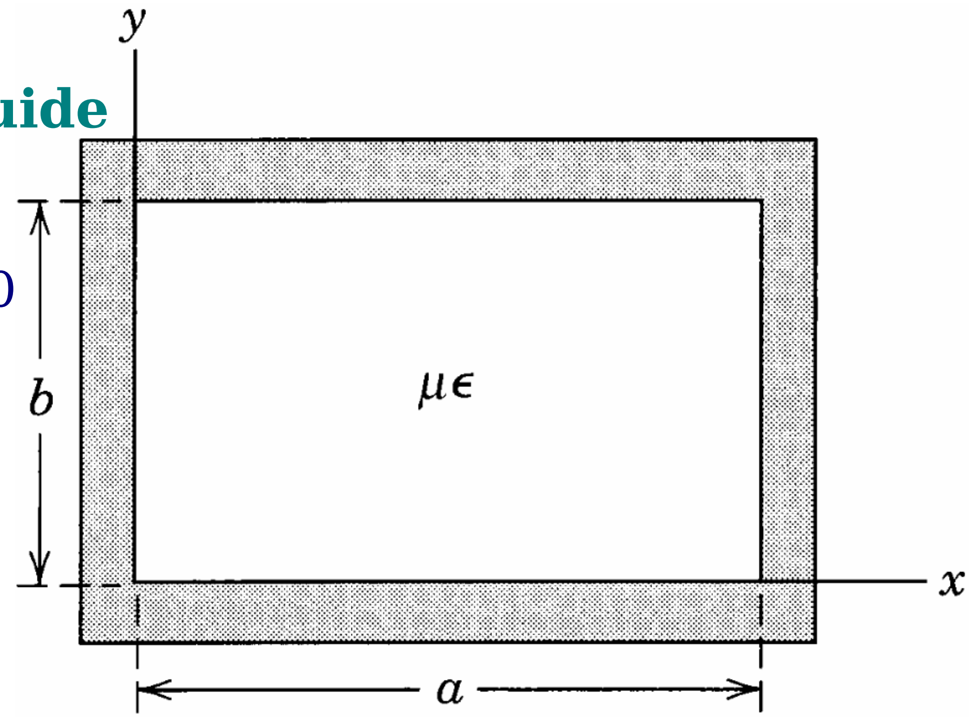
- at any given frequency only a finite number of modes can propagate.

- It is often convenient to choose the dimensions of the guide so that at the operating frequency only the lowest mode can occur.

- $k_\lambda > k_0 = \sqrt{\mu\,\epsilon}\,\omega \quad \Rightarrow \quad$ phase velocity $\quad v_p = \dfrac{\omega}{k_\lambda} = \dfrac{1}{\sqrt{\mu\,\epsilon\,(1-\omega_\lambda^2/\omega^2)}} > \dfrac{1}{\sqrt{\mu\,\epsilon}}$

The phase velocity becomes infinite exactly at cutoff.

8.4 Modes in a Rectangular Waveguide



- For a TE wave

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\gamma^2\right)\psi=0 \quad \Leftarrow \quad \psi=H_z, \quad \frac{\partial \psi}{\partial n}=0$$

$$\Rightarrow \quad \psi_{mn}(x,y)=H_0\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$\text{and} \quad \gamma_{mn}^2=\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$$

$$\Rightarrow \quad \text{the cutoff freqency} \quad \omega_{mn}=\frac{\pi}{\sqrt{\mu\epsilon}}\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}} \quad \Rightarrow \quad \omega_{1,0}=\frac{\pi}{a\sqrt{\mu\epsilon}}$$

the lowest cutoff freq. for $a>b$

$$\Rightarrow \quad H_z=H_0\cos\frac{\pi x}{a}e^{i(kz-\omega t)}, \quad H_x=-i\frac{ka}{\pi}H_0\sin\frac{\pi x}{a}e^{i(kz-\omega t)} \quad \Leftarrow \quad \text{for } \mathrm{TE}_{1,0} \text{ mode}$$

$$E_y=i\frac{\omega\mu a}{\pi}H_0\sin\frac{\pi x}{a}e^{i(kz-\omega t)} \quad \Leftarrow \quad k=k_{1,0}$$

- The presence of a factor i in H_x (E_y) means that there is a spatial (or temporal) phase difference of 90° between H_x (E_y) and H_z in the propagation direction.

- the $\mathrm{TE}_{1,0}$ mode has the lowest cutoff frequency of both TE and TM modes, and so is the one used in most practical situations.

ω_{mn} for $a = 2b$

$m \backslash n$	0	1	2	3
0		2.00	4.00	6.00
1	1.00	2.24	4.13	
2	2.00	2.84	4.48	
3	3.00	3.61	5.00	
4	4.00	4.48	5.66	
5	5.00	5.39		
6	6.00			

- There is a frequency range from cutoff to twice cutoff or to a/b times cutoff where the $TE_{1,0}$ mode is the only propagating mode.

8.5 Energy Flow and Attenuation in Waveguides

- $$\mathbf{S} = \frac{1}{2}\,\mathbf{E}\times\mathbf{H}^* = \frac{\omega\,k}{2\,\gamma^4}\left[\begin{array}{ll} \epsilon\,[\,\hat{\mathbf{z}}\,|\nabla_t\,\psi|^2 + i\,\dfrac{\gamma^2}{k}\,\psi\,\nabla_t\,\psi^*\,] & (\mathrm{TM}) \\ \mu\,[\,\hat{\mathbf{z}}\,|\nabla_t\,\psi|^2 - i\,\dfrac{\gamma^2}{k}\,\psi^*\,\nabla_t\,\psi\,] & (\mathrm{TE}) \end{array}\right. \qquad \Leftarrow \quad (3)\ \ \&\ \ (4)$$

$\Rightarrow$ $\mathbf{S}_t$: reactive energy flow $\qquad \Leftarrow \quad \psi \in \mathbb{R}$

S_z : time-averaged flow of energy

$$\Rightarrow \quad P = \int_A \mathbf{S}\cdot\hat{\mathbf{z}}\;\mathrm{d}\,a = \frac{\omega\,k}{2\,\gamma^4}\begin{bmatrix}\epsilon\\ \mu\end{bmatrix}\int_A (\nabla_t\,\psi)^*\cdot\nabla_t\,\psi\;\mathrm{d}\,a$$

$$= \frac{\omega\,k}{2\,\gamma^4}\begin{bmatrix}\epsilon\\ \mu\end{bmatrix}\left[\cancel{\oint_C \psi^*\,\frac{\partial\,\psi}{\partial\,n}\,\mathrm{d}\,\ell} - \int_A \psi^*\,\nabla_t^2\,\psi\;\mathrm{d}\,a\right]$$

$$\Rightarrow \quad P = \frac{1}{2\sqrt{\mu\,\epsilon}}\,\frac{\omega^2}{\omega_\lambda^2}\sqrt{1-\frac{\omega^2}{\omega_\lambda^2}}\begin{bmatrix}\epsilon\\ \mu\end{bmatrix}\int_A \psi^*\,\psi\;\mathrm{d}\,a\,, \quad \text{similarly} \quad U = \frac{1}{2}\,\frac{\omega^2}{\omega_\lambda^2}\begin{bmatrix}\epsilon\\ \mu\end{bmatrix}\int_A \psi^*\,\psi\;\mathrm{d}\,a$$

$$\Rightarrow \quad \frac{P}{U} = \frac{k}{\omega}\,\frac{1}{\mu\,\epsilon} = \frac{1}{\sqrt{\mu\,\epsilon}}\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}} = v_g = \frac{\mathrm{d}\,\omega}{\mathrm{d}\,k}$$

group velocity $\quad \Rightarrow$

$$v_g < v_p\,, \quad v_g\,(\omega_\lambda) = 0$$

$$v_g\,v_p = \frac{1}{\mu\,\epsilon} \quad \Rightarrow \quad \omega\;\mathrm{d}\,\omega \propto k\;\mathrm{d}\,k$$

- If the walls have a finite conductivity, there is ohmic losses and the power flow along the guide will be attenuated.

$$\alpha_\lambda : \text{unimportant except near cutoff when } k_\lambda^{(0)} \to 0$$

$$k_\lambda \simeq k_\lambda^{(0)} + \alpha_\lambda + i\,\beta_\lambda \quad \Leftarrow \quad \beta_\lambda = -\frac{1}{2P}\frac{\mathrm{d}P}{\mathrm{d}z} \quad \Leftarrow \quad P = P_0\, e^{-2\beta_\lambda z}$$

$$-\frac{\mathrm{d}P}{\mathrm{d}z} = \frac{1}{2\sigma\delta}\oint_C |\mathbf{n}\times\mathbf{H}|^2\,\mathrm{d}\ell = \frac{1}{2\sigma\delta}\left[\begin{array}{ll} \displaystyle\oint_C \frac{\omega^2}{\mu^2\omega_\lambda^4}\left|\frac{\partial\psi}{\partial n}\right|^2 \mathrm{d}\ell & (\mathrm{TM}) \\ \displaystyle\oint_C \left[\frac{\omega^2-\omega_\lambda^2}{\mu\,\epsilon\,\omega_\lambda^4}|\mathbf{n}\times\nabla_t\psi|^2 + |\psi|^2\right]\mathrm{d}\ell & (\mathrm{TE}) \end{array}\right. \qquad (11)$$

$$\left\langle\left|\frac{\partial\psi}{\partial n}\right|^2\right\rangle \sim \langle|\mathbf{n}\times\nabla_t\psi|^2\rangle \sim \mu\,\epsilon\,\omega_\lambda^2\langle|\psi|^2\rangle \quad \Leftarrow \quad (\nabla_t^2 + \mu\,\epsilon\,\omega_\lambda^2)\,\psi = 0$$

$$\Rightarrow \quad \oint_C \left|\frac{\partial\psi}{\partial n}\right|^2 \frac{\mathrm{d}\ell}{\omega_\lambda^2} = \xi_\lambda\,\mu\,\epsilon\,\frac{C}{A}\int_A |\psi|^2\,\mathrm{d}a$$

$$\Rightarrow \quad \beta_\lambda = \sqrt{\frac{\epsilon}{\mu}}\,\frac{1}{\sigma\,\delta_\lambda}\,\frac{C}{2A}\sqrt{\frac{\omega/\omega_\lambda}{1-\omega_\lambda^2/\omega^2}}\left[\xi_\lambda + \eta_\lambda\frac{\omega_\lambda^2}{\omega^2}\right] \quad (5) \quad \Leftarrow \quad \delta_\lambda \equiv \sqrt{\frac{2}{\mu\,\sigma\,\omega_\lambda}}, \quad \text{For TM} \quad \eta_\lambda = 0$$

- For the TE modes in a rectangular guide,

$$\xi_{m,0} = \frac{a}{a+b}, \quad \eta_{m,0} = \frac{2b}{a+b} \quad \Leftarrow \quad \text{order of unity}$$

- For TM modes the minimum always occurs at

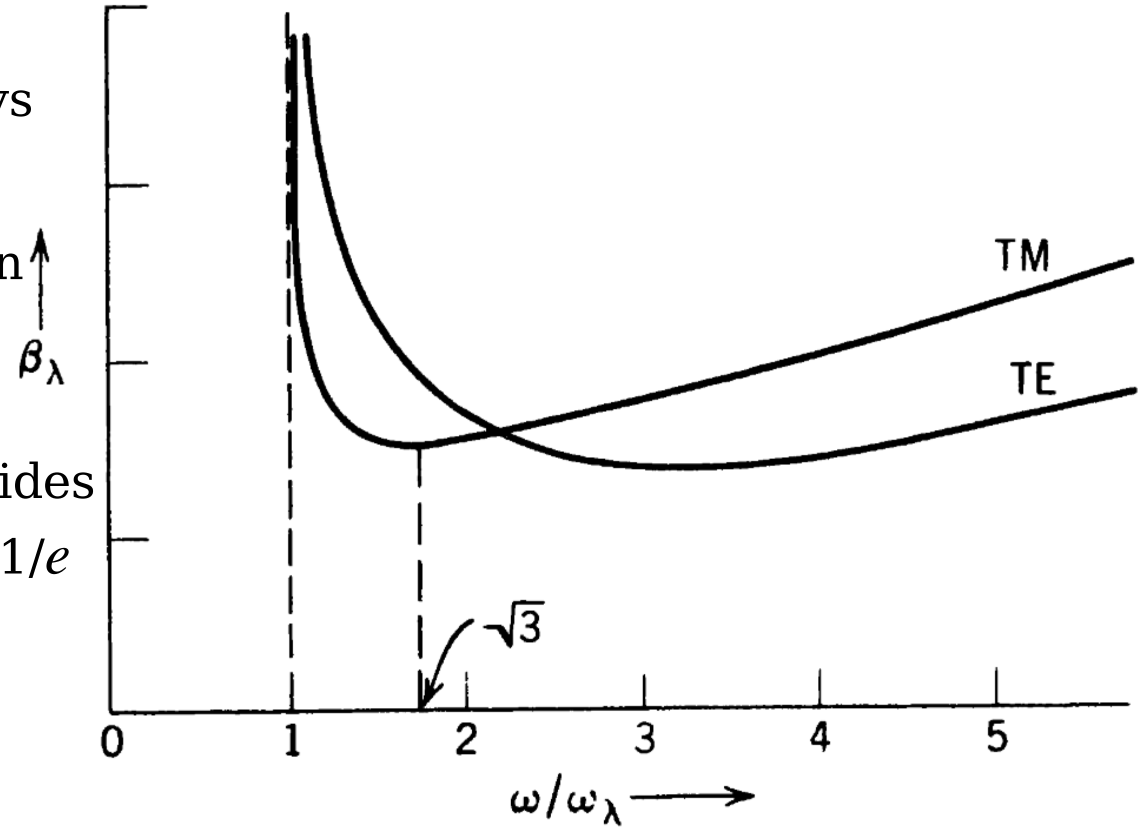
$$\omega_{\min} = \sqrt{3}\,\omega_\lambda$$

- At high frequencies the attenuation increases as $\omega^{1/2}$.

- In the microwave region typical attenuation constants for copper guides are of the order $\beta_\lambda \sim 10^{-4}\omega_\lambda/c$, giving $1/e$ distances of 200-400 meters.

- (5) break down close to cutoff for

$$\beta_\lambda \to \infty \quad \text{at} \quad \omega = \omega_\lambda$$



8.6 Perturbation of Boundary Conditions

- The use of energy conservation can determine the attenuation constant β_λ, but gives physically meaningless results at cutoff and fails to yield a value for α_λ.

—can be obtained by use of the *perturbation of boundary conditions*.

- consider a single TM mode with no other mode degenerate with it

$$(\nabla_t^2+\gamma_0^2)\,\psi_0=0,\quad \psi_{0\,|S}=0,\quad \gamma_0^2\in\mathbb{R}\quad \Leftarrow\quad E_z=\psi_0\quad \text{unperturbed}$$

$$(4)\ \&\ (6)\quad\Rightarrow\quad \psi_{|S}\simeq f\,\frac{\partial\psi_0}{\partial n}\Big|_S\quad\Leftarrow\quad f=(1+i)\,\frac{\mu_c\,\delta}{2\,\mu}\,\frac{\omega^2}{\omega_0^2}\quad\Leftarrow\quad \omega_0:\ \text{cutoff freq. of the unperturbed mode}$$

$$\Rightarrow\quad \text{the perturbed problem}\quad (\nabla_t^2+\gamma^2)\,\psi=0,\quad \psi_{|S}\simeq f\,\partial\psi_0/\partial n_{|S}\quad (7)$$

$$\int_A[\phi\,\nabla_t^2\psi-\psi\,\nabla_t\phi]\,\mathrm{d}a=\oint_C[\psi\,\frac{\partial\phi}{\partial n}-\phi\,\frac{\partial\psi}{\partial n}]\,\mathrm{d}\ell\quad \text{2d Green's theorem}\quad +\quad \phi=\psi_0^*$$

$$\Rightarrow\quad (\gamma_0^2-\gamma^2)\int_A\psi_0^*\,\psi\,\mathrm{d}a=f\oint_C\left|\frac{\partial\psi_0}{\partial n}\right|^2\mathrm{d}\ell\,,\quad k^{(0)}\,\beta_{\mathrm{TM}}^{(0)}\int_A|\psi_0|^2\,\mathrm{d}a=\frac{\mu_c\,\delta\,\omega^2}{4\,\mu\,\omega_0^2}\oint_C\left|\frac{\partial\psi_0}{\partial n}\right|^2\mathrm{d}\ell$$

$$\Rightarrow\quad \gamma_0^2-\gamma^2=k^2-k^{(0)\,2}\simeq f\,\frac{\oint_C\left|\frac{\partial\psi_0}{\partial n}\right|^2\mathrm{d}\ell}{\int_A|\psi_0|^2\,\mathrm{d}a}\quad\Rightarrow\quad k^2\simeq k^{(0)\,2}+2\,(1+i)\,k^{(0)}\,\beta^{(0)}\quad (8)$$

for both the TM/TE mode

$$k^{(0)} \gg \beta^{(0)} \quad \Rightarrow \quad k \simeq k^{(0)} + \alpha^{(0)} + i\,\beta^{(0)} \quad \Leftarrow \quad \alpha^{(0)} = \beta^{(0)}$$

- At cutoff and below where the earlier results failed, (8) yields sensible results because $k^{(0)}\beta^{(0)}$ is finite and well behaved in the neighborhood of $k^{(0)}=0$.

- The transition from a propagating mode to a cutoff mode is not a sharp one if the walls are less than perfect conductors, but the attenuation is sufficiently large immediately above & below the cutoff frequency that little error is made in assuming a sharp cutoff.

- If a TM and a TE mode are degenerate, then any perturbation can cause sizable mixing of the two modes. The methods used so far fail in such circumstances.

- The breakdown of this method occurs in the perturbed boundary condition (7), now involving the tangential derivative of H_z and the normal derivative of E_z.

- The perturbed modes are orthogonal linear combinations of the unperturbed TM and TE modes

$$\beta = \frac{1}{2}(\beta_{\mathrm{TM}} + \beta_{\mathrm{TE}}) \pm \frac{1}{2}\sqrt{(\beta_{\mathrm{TM}} - \beta_{\mathrm{TE}})^2 + 4\,|K|^2} \quad \Leftarrow \quad K : \text{coupling parameter}$$

8.7 **Resonant Cavities**

- an important class of cavities is produced by placing end faces on a length of cylindrical waveguide—the end surfaces are plane ⊥ to the axis of the cylinder.

- Because of reflections at the end surfaces, the z dependence of the fields is that appropriate to standing waves: $A \sin kz + B \cos kz$

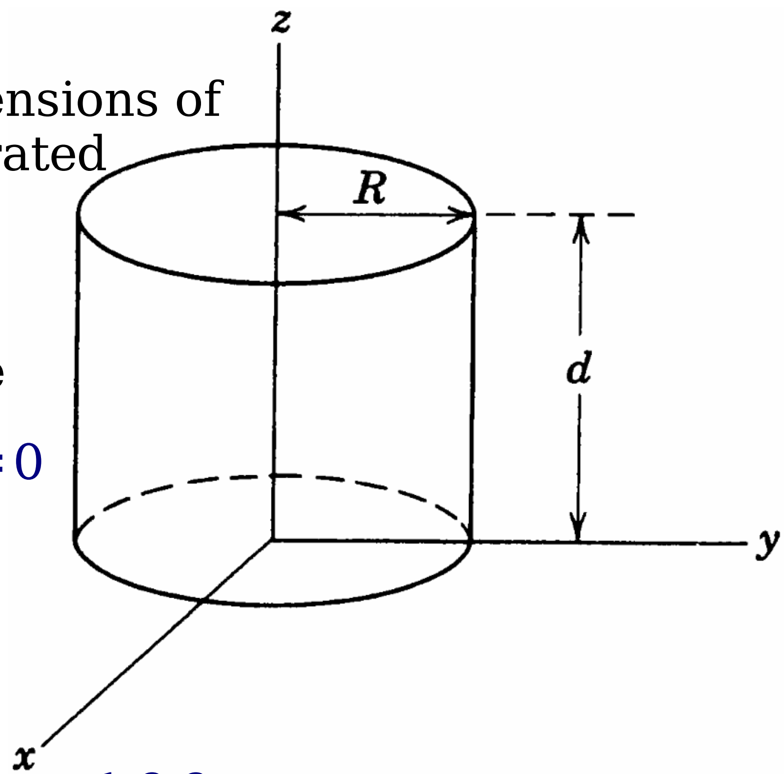
- $z \in [0, d] \quad \Rightarrow \quad k = p\dfrac{\pi}{d}, \quad p = 0, 1, 2, \cdots$

$$\Rightarrow \begin{array}{l} \mathbf{E}_t(z=0) = \mathbf{E}_t(z=d) = 0 \quad \Rightarrow \quad E_z = \psi(x, y)\cos\dfrac{p\pi z}{d} \quad (\mathrm{TM}) \\ H_z(z=0) = H_z(z=d) = 0 \quad \Rightarrow \quad H_z = \psi(x, y)\sin\dfrac{p\pi z}{d} \quad (\mathrm{TE}) \end{array} \tag{9}$$

$$\Rightarrow \begin{array}{l} \mathbf{E}_t = -\dfrac{p\pi}{d\gamma^2}\sin\dfrac{p\pi z}{d}\nabla_t\psi, \quad \mathbf{H}_t = \dfrac{i\epsilon\omega}{\gamma^2}\cos\dfrac{p\pi z}{d}\hat{\mathbf{z}}\times\nabla_t\psi \quad (\mathrm{TM}) \\ \mathbf{E}_t = -\dfrac{i\mu\omega}{\gamma^2}\sin\dfrac{p\pi z}{d}\hat{\mathbf{z}}\times\nabla_t\psi, \quad \mathbf{H}_t = \dfrac{p\pi}{d\gamma^2}\cos\dfrac{p\pi z}{d}\nabla_t\psi \quad (\mathrm{TE}) \end{array} \tag{10}$$

where $\gamma^2 = \mu\epsilon\omega^2 - \dfrac{p^2\pi^2}{d^2} \quad \Rightarrow \quad \omega_{\lambda p}^2 = \dfrac{1}{\mu\epsilon}(\gamma_\lambda^2 + k_p^2) = \dfrac{1}{\mu\epsilon}\left[\gamma_\lambda^2 + \dfrac{p^2\pi^2}{d^2}\right]$

- It is usually expedient to choose the various dimensions of the cavity so that the resonant freq. lies well separated from other resonant freq. and the cavity will be stable and insensitive to perturbing effects.



- Consider a right circular cylinder, for a TM mode

$$\psi = E_z , \quad E_z(\rho = R) = 0 \quad \searrow \quad x_{mn} : n^{\text{th}} \text{ root of } J_m(x) = 0$$

$$\Rightarrow \quad \psi(\rho, \phi) = E_0 J_m(\gamma_{mn} \rho)\, e^{\pm i m \phi} \quad \Leftarrow \quad \gamma_{mn} = \frac{x_{mn}}{R}$$

$$\Rightarrow \quad \omega_{mnp} = \frac{1}{\sqrt{\mu \epsilon}} \sqrt{\frac{x_{mn}^2}{R^2} + \frac{p^2 \pi^2}{d^2}} \quad \Leftarrow \quad m : 0,1,2,\cdots, \quad n : 1,2,3,\cdots$$

$$\Rightarrow \quad \text{the lowest TM mode } \mathrm{TM}_{0,1,0} \text{ has } m = 0,\ n = 1,\ p = 0 \quad \Rightarrow \quad \omega_{010} = \frac{2.405}{\sqrt{\mu \epsilon}\, R}$$

$$\Rightarrow \quad E_z = E_0 J_0\left(\frac{2.405\, \rho}{R}\right) e^{-i \omega t}, \quad H_\phi = -i \sqrt{\frac{\epsilon}{\mu}}\, E_0 J_1\left(\frac{2.405\, \rho}{R}\right) e^{-i \omega t}$$

- The resonant freq. for this mode is indep. of d. So simple tuning is impossible.

- For TE modes, $\psi = H_z , \quad \frac{\partial \psi}{\partial \rho}\Big|_R = 0 \quad \Rightarrow \quad \psi(\rho, \phi) = E_0 J_m(\gamma_{mn} \rho)\, e^{\pm i m \phi} \quad \Leftarrow \quad \gamma_{mn} = \frac{x'_{mn}}{R}$

$$x'_{mn} : n^{\text{th}} \text{ root of } J'_m(x) = 0$$

- $\omega_{mnp} = \dfrac{1}{\sqrt{\mu\,\epsilon}} \sqrt{\dfrac{x'^2_{mn}}{R^2} + \dfrac{p^2\,\pi^2}{d^2}} \quad \Leftarrow \quad m: 0,1,2,\cdots, \quad n\,,\ p: 1,2,3,\cdots$

$$\Rightarrow \quad \text{the lowest TE mode } \mathrm{TE}_{1,1,1} \text{ has } m=1,\ n=1,\ p=1 \quad \Rightarrow \quad \omega_{111} = \frac{1.841}{\sqrt{\mu\,\epsilon}\,R} \sqrt{1 + 2.912\,\frac{R^2}{d^2}}$$

$$\Rightarrow \quad \psi = H_z = H_0\, J_1\left(\frac{1.841\,\rho}{R}\right) \cos\phi\, \sin\frac{\pi\,z}{d}\, e^{-i\,\omega\,t} \quad \Leftarrow \quad (10)$$

- For $d > 2.03R$, the resonance frequency ω_{111} is smaller than that for the lowest TM mode. Then the $\mathrm{TE}_{1,1,1}$ mode is the fundamental oscillation of the cavity.

- Because the frequency depends on the ratio d/R it is possible to provide easy tuning by making the separation of the end faces adjustable.

8.8 **Power Losses in a Cavity; Q of a Cavity**

- If one attempts to excite a particular mode in a cavity, no fields of the right sort could be built up unless the exciting frequency were exactly equal to the chosen resonance frequency.

- In fact there will not be a delta function singularity, but rather a narrow band of frequencies around the eigenfrequency over which excitation can occur.

- An important source of this smearing out of the sharp frequency is the energy dissipation in the cavity walls and/or in the dielectric filling the cavity.

- A measure of the sharpness of response of the cavity to external excitation is

$$Q = \frac{2\pi \times \text{the ratio of the time-averaged energy}}{\text{stored in the cavity to the energy loss per cycle}} = \omega_0 \frac{\text{Stored energy}}{\text{Power loss}} \Leftarrow \quad \omega_0 : \text{resonance frequency}$$

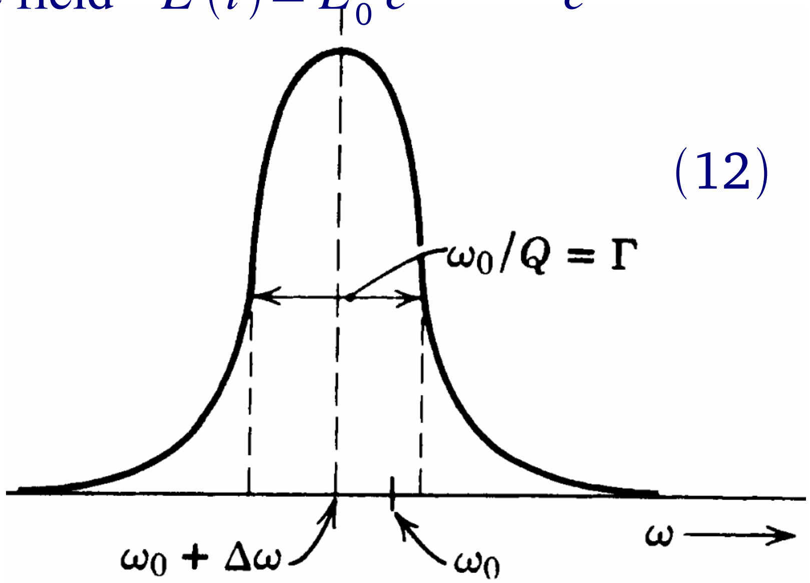
$$\Rightarrow \quad \frac{dU}{dt} = -\frac{\omega_0}{Q} U \quad \Rightarrow \quad U(t) = U_0 e^{-\omega_0 t/Q} \quad \Rightarrow \quad \text{the field} \quad E(t) = E_0 e^{-\omega_0 t/2Q} e^{-i(\omega_0 + \Delta\omega)t}$$

$$E(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E(\omega) e^{-i\omega t} d\omega \qquad (12)$$

where

$$E(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} E_0 e^{-\omega_0 t/2Q} e^{i(\omega - \omega_0 - \Delta\omega)t} dt$$

$$\Rightarrow \quad |E(\omega)|^2 \propto \frac{1}{(\omega - \omega_0 - \Delta\omega)^2 + (\omega_0/2Q)^2}$$



- the frequency separation between half-power points $Q = \dfrac{\omega_0}{\delta\,\omega} = \dfrac{\omega_0}{\Gamma}$
- To determine the Q of a cavity, consider only the cylindrical cavities

$$U = \frac{d}{4}\begin{bmatrix}\epsilon\\ \mu\end{bmatrix}\left[1+\left(\frac{p\,\pi}{\gamma_\lambda\,d}\right)^2\right]\int_A |\psi|^2\,\mathrm{d}\,a \;\Leftarrow\; \begin{matrix}\text{TM} & \Leftarrow\; U = \text{the result}\times 2 \quad \text{for} \quad p=0 \\ \text{TE} & \end{matrix} \;\Leftarrow\; \begin{matrix}(9)\\(10)\end{matrix}$$

$$\Rightarrow \quad P_{\text{loss}} = \frac{1}{2\,\sigma\,\delta}\left[\oint_C \int_0^d |\mathbf{n}\times\mathbf{H}|^2_{\text{sides}}\,\mathrm{d}\,z\,\mathrm{d}\,\ell + 2\int_A |\mathbf{n}\times\mathbf{H}|^2_{\text{sides}}\,\mathrm{d}\,a\right] \quad \Leftarrow \quad (11)$$

$$\Rightarrow \quad P_{\text{loss}} = \frac{\epsilon}{\sigma\,\delta\,\mu}\left[1+\left(\frac{p\,\pi}{\gamma_\lambda\,d}\right)^2\right]\begin{bmatrix}\left[1+\ \xi_\lambda\dfrac{C\,d}{4\,A}\right]\displaystyle\int_A |\psi|^2\,\mathrm{d}\,a \quad \text{for} \quad p\neq 0\\ \left[1+2\,\xi_\lambda\dfrac{C\,d}{4\,A}\right]\displaystyle\int_A |\psi|^2\,\mathrm{d}\,a \quad \text{for} \quad p=0\end{bmatrix} \quad \text{of TM modes}$$

$$\Rightarrow \quad Q = \frac{\mu}{\mu_c}\begin{bmatrix}\dfrac{d}{2\,\delta}\left(1+\xi_\lambda\dfrac{C\,d}{4\,A}\right)^{-1} \quad \text{for} \quad p\neq 0\\ \dfrac{d}{\delta}\left(1+\xi_\lambda\dfrac{C\,d}{2\,A}\right)^{-1} \quad \text{for} \quad p=0\end{bmatrix} \quad \text{of TM modes}$$

$$\Rightarrow \quad \begin{matrix}\text{physical}\\ \text{interpretation}\end{matrix} \quad Q = \frac{\mu}{\mu_c}\,\frac{V}{S\,\delta}\times(\text{Geometrical factor}) \quad \Leftarrow \quad \begin{matrix}V:\text{ volume of the cavity}\\ S:\text{ total surface area}\end{matrix}$$

- The Q of a cavity is the ratio of the volume occupied by the fields to the volume of the conductor into which the fields penetrate because of the finite conductivity.

- The expression for Q applies to cavities of arbitrary shape, with an appropriate geometrical factor of the order of unity.

- For the $\mathrm{TE}_{1,1,1}$ mode in the right circular cylinder cavity,

$$\text{geometrical factor} = \left(1+\frac{d}{R}\right)\frac{1+0.343\, d^2/R^2}{1+0.209\, d/R+0.244\, d^3/R^3} \quad \begin{matrix} \in [1,\ 2.13\ (\max),\ 1.42] \\ \text{for } \ d/R \in [0,\ 1.91,\ \infty] \end{matrix}$$

- Possible shifts in frequency cannot be calculated with the energy conservation, but the perturbation of boundary conditions again removes these deficiencies. With the similar procedures in Sec. 8.6

$$\omega_0^2-\omega^2 \simeq (1+i)\, I \quad \Leftarrow \quad I : \text{ratio of appropriate integrals} \quad \Rightarrow \quad \Im[\omega] = -\frac{I}{2\,\omega_0} \quad \text{for} \quad I \to 0$$

$$(12) \quad \Rightarrow \quad \Im[\omega] = -\frac{\omega_0}{2\,Q} \quad \Rightarrow \quad I = \frac{\omega_0^2}{Q} \quad \Rightarrow \quad \omega^2 \simeq \omega_0^2\left[1-\frac{1+i}{Q}\right]$$

Damping causes equal modifications to the real and imaginary parts of ω^2

$$\text{For large } \ Q \quad \Rightarrow \quad \Delta\omega \simeq \Im[\omega] \simeq -\frac{\omega}{2\,Q} \quad \Leftarrow$$

The resonant frequency is always lowered by the presence of resistive losses

- The near equality of the real and imaginary parts of the change in ω^2 is a consequence of the boundary condition (13) appropriate for good conductors.

- It is possible for other system the relative magnitude of the real and imaginary parts of the change in ω^2 can be different.

8.9 Earth & Ionosphere as a Resonant Cavity: Schumann Resonances

- An example of a resonant cavity is the earth as one boundary surface and the ionosphere as the other.

- The lowest resonant modes are of very low frequency, λ~the earth's radius.

- Seawater ($\sigma \sim 0.1\Omega^{-1}\text{m}^{-1}$) & ionosphere ($\sigma < 10^{-4}\Omega^{-1}\text{m}^{-1}$) are not perfect conducting.

- idealize the physical reality and consider as a model two perfectly conducting, concentric spheres with radii a(=6400km, earth's radius) and $b=a+h$ (h=100km, the ionosphere's height.)

- focus our attention on the TM modes for the lowest freq., with only tangential **B**.

- the TM modes, with a radially directed **E**, can satisfy the boundary condition of vanishing tangential **E** at boundaries without radial variation of the fields.

$\Rightarrow$ the freq. for the lowest TM modes $\omega_{\text{TM}} \sim c/a$.

- the TE modes, with only tangential **E**, have a radial variation about half a wavelength between the boundaries.

$\Rightarrow$ The lowest freq. for the TE modes $\omega_{\text{TE}} \sim \pi c/h$.

- $$\frac{\partial \mathbf{B}}{\partial \phi} = 0, \quad B_r = 0 \quad \Rightarrow \quad B_\theta = 0, \quad B_\phi \neq 0 \quad \Leftarrow \quad \nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad E_\phi = 0 \quad \Leftarrow \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\Rightarrow \quad B_\phi \neq 0, \quad E_r \neq 0, \quad E_\theta \neq 0 \quad \text{independent of } \phi$$

- Maxwell's eqns $\Rightarrow \quad \frac{\omega^2}{c^2}\mathbf{B} - \nabla\times\nabla\times\mathbf{B} = 0$

$$\Rightarrow \quad \frac{\omega^2}{c^2} r B_\phi + \frac{\partial^2}{\partial r^2}(r B_\phi) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\partial\theta (r B_\phi \sin\theta)\right] = 0 \quad \Leftarrow \quad \phi \text{ component}$$

$$\Rightarrow \quad \frac{\omega^2}{c^2} r B_\phi + \frac{\partial^2}{\partial r^2}(r B_\phi) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial r B_\phi}{\partial\theta}\right) - \frac{r B_\phi}{\sin^2\theta}\right] = 0 \quad \Rightarrow \quad m = \pm 1$$

$$\Rightarrow \quad B_\phi(r,\theta) = \frac{u_\ell(r)}{r} p_\ell^1(\cos\theta) \quad \Rightarrow \quad \frac{d^2 u_\ell}{d r^2} + \left[\frac{\omega^2}{c^2} - \frac{\ell(\ell+1)}{r^2}\right] u_\ell = 0$$

$$E_r = \frac{i c^2}{\omega r \sin\theta}\frac{\partial}{\partial\theta}(B_\phi \sin\theta) = -\frac{i c^2}{\omega r}\ell(\ell+1)\frac{u_\ell}{r} p_\ell(\cos\theta) \qquad \Downarrow \quad E_\theta(a) = E_\theta(b) = 0$$

$$E_\theta = -\frac{i c^2}{\omega r}\frac{\partial}{\partial r}(r B_\phi) \qquad = -\frac{i c^2}{\omega r}\frac{d u_\ell}{d r} p_\ell^1(\cos\theta) \quad \Rightarrow \quad \frac{d u_\ell(a)}{d r} = \frac{d u_\ell(b)}{d r} = 0 \quad (14)$$

$$h/a \ll 1 \quad \Rightarrow \quad \frac{\ell(\ell+1)}{r^2} \simeq \frac{\ell(\ell+1)}{a^2} \quad \Rightarrow \quad u_\ell \simeq \left|\begin{matrix}\sin(q r)\\ \cos(q r)\end{matrix}\right. \quad \Leftarrow \quad q^2 = \frac{\omega^2}{c^2} - \frac{\ell(\ell+1)}{a^2}$$

$$\Rightarrow \quad u_\ell \simeq A\cos[q(r-a)] \quad \text{where} \quad q h = n\pi, \quad n = 0, 1, 2, \cdots \quad \Leftarrow \quad (14)$$

- For $n>0$ the freq. of the modes are larger than $\omega = n\pi c/h$ and are in the domain of freq of the TE modes. Only for $n=0$ are there very-low-frequency modes.

$$q = 0 \quad \Rightarrow \quad u_\ell = \text{const} \quad \Rightarrow \quad \omega_\ell \simeq \frac{c}{a}\sqrt{\ell(\ell+1)} \quad \text{Schumann resonances} \quad \Leftarrow \quad \frac{h}{a} \to 0$$

- To 1st-order in h/a the correct result has a replaced by $a+h/2$.

- $$E_\theta = 0, \quad E_r \propto \frac{P_\ell(\cos\theta)}{r^2}, \quad B_\phi \propto \frac{P_\ell^1(\cos\theta)}{r}$$

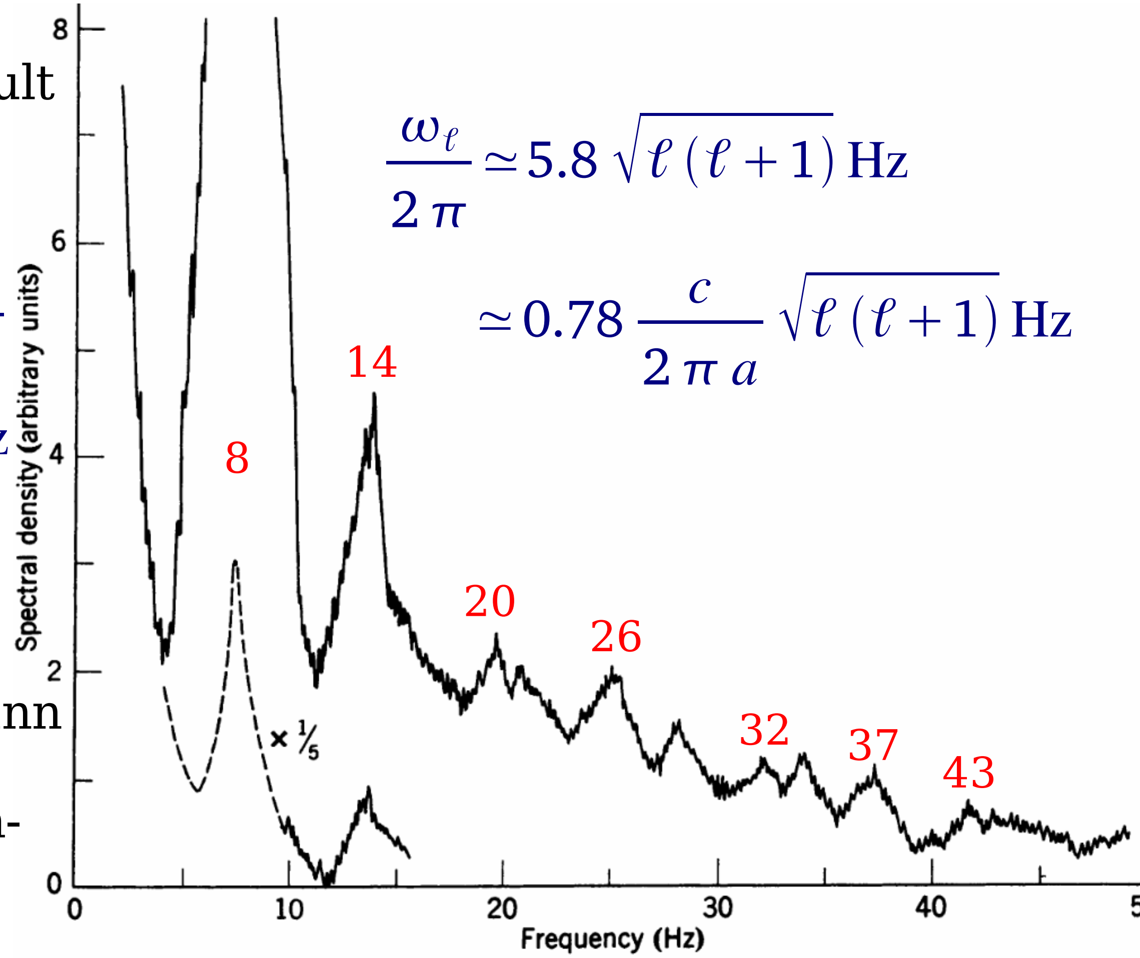
- $$\frac{\omega_\ell}{2\pi} = 10.6,\ 18.3,\ 25.8,\ 33.4,\ 40.9 \text{ Hz}$$

- Lightning bolts act as sources of radial electric fields.

- The frequencies near the Schumann resonances are preferred because they are normal modes of the earth-ionosphere cavity.

$$\frac{\omega_\ell}{2\pi} \simeq 5.8\sqrt{\ell(\ell+1)}\ \text{Hz}$$

$$\simeq 0.78\,\frac{c}{2\pi a}\sqrt{\ell(\ell+1)}\ \text{Hz}$$



- The lack of precise agreement is that the assumption of perfectly conducting walls is rather far from the truth.

- Q~4-10 for the first few resonances, corresponding to rather heavy damping.

- 1st curiosity: A nuclear explosion can decease 3-5% in Schumann resonant freq. by the alterations in the ionosphere.

- 2nd curiosity: Schumann resonances can serve as a global tropical thermometer, due to the Schumann resonant **B** depends on the freq. of lightning, and the freq. of lightning depends on temperature.

8.10 Multimode Propagation in Optical Fibers

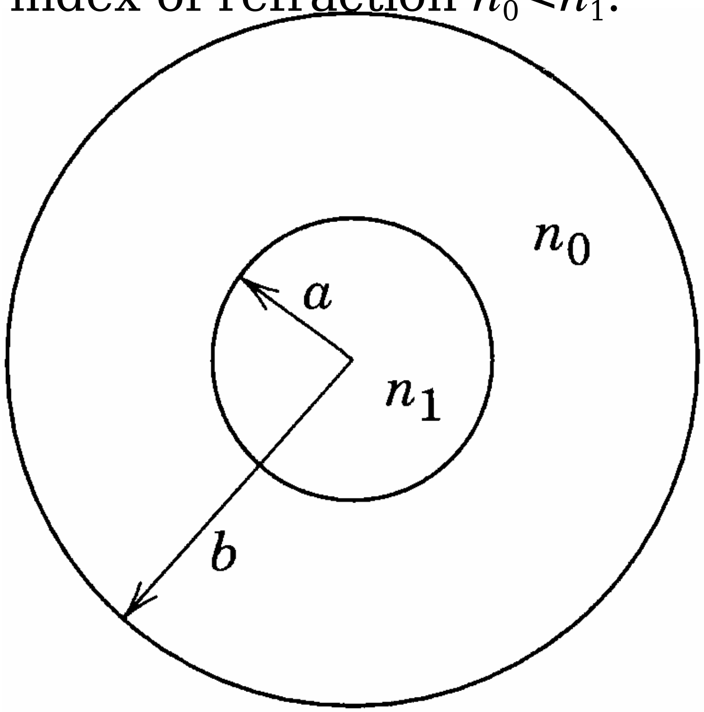
- Transmission via optical fibers falls approximately into two classes:

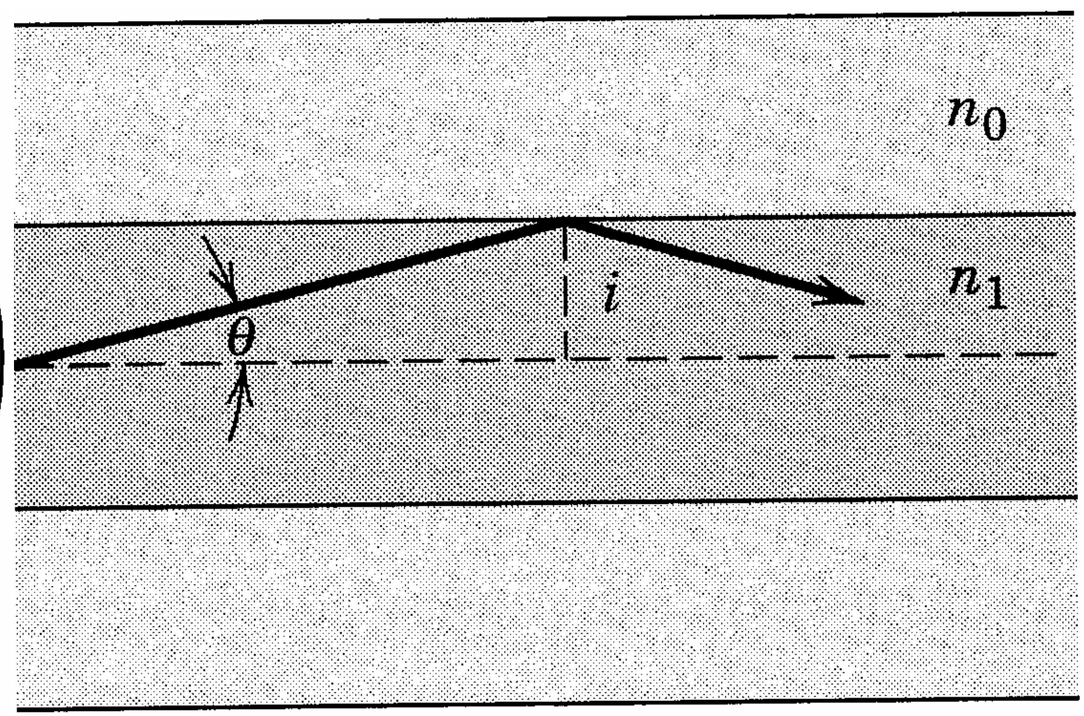
(1) multimode: cores are typically 50μm in diameter for a wavelength ~ 1μm;
(2) single-mode: cores are around 5μm diameters.

- consider multimode transmission with the semi-geometrical eikonal or WKB.

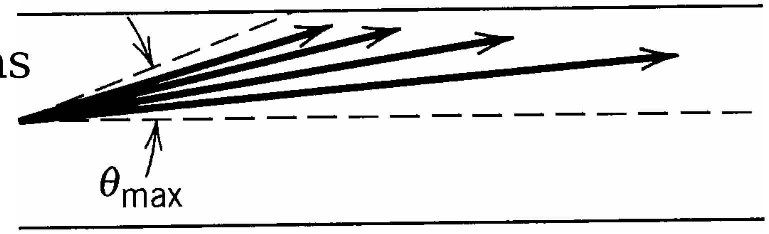
- Optical fiber cables, ~2cm in diameter, are actually nests of smaller cables with 6 or 8 optical fibers protected by secondary coatings and buffer layers.

- The operative fiber consists of a cylindrical core of radius a [$2a$=O(50μm)] and index of refraction n_1 surrounded by a cladding of outer radius b [$2b$=O(150μm)] and index of refraction n_0<n_1.





- Since the wavelength of the light ~ 1 μm, the ideas of geometrical optics apply; the interface between core and cladding can be treated as locally flat.



- If the angle of incidence i of a ray is greater than i_0 ($i_0=\sin^{-1}(n_0/n_1)$, the critical angle for total internal reflection), the ray will be confined and thus propagate.

Propagation occurs for rays with $\theta < \theta_{\max} = \cos^{-1}\dfrac{n_0}{n_1}$

- Define $\Delta \equiv \dfrac{n_1^2 - n_0^2}{2\, n_1^2} \approx 1 - \dfrac{n_0}{n_1} \quad \Rightarrow \quad \Delta \le 1\,\%$ typically $\quad \Rightarrow \quad \theta_{\max} \approx \sqrt{2\,\Delta} \le 0.14 = 8^{\mathrm{o}}$

- 2d phase-space number density $\quad \mathrm{d}\,N = \pi\, a^2 \dfrac{\mathrm{d}^2 k}{(2\,\pi)^2} \cdot 2 = \text{spatial area} \times \dfrac{\text{wave number}}{\text{volume elememt}} \times 2$ polarizations

$$k_\perp \approx k\,\theta < k\,\theta_{\max} \quad \Rightarrow \quad \mathrm{d}^2 k = 2\,\pi\, k_\perp\, \mathrm{d}\,k_\perp = 2\,\pi\, k^2\,\theta\,\mathrm{d}\,\theta$$

$$\Rightarrow \quad N \approx a^2\, k^2 \int_0^{\theta_{\max}} \theta\,\mathrm{d}\,\theta \approx \frac{1}{2}\, V^2 \quad \Leftarrow \quad V \equiv k\,a\sqrt{2\,\Delta} \quad \text{fiber parameter}$$

Typically $\lambda = 0.85\,\mu\,\mathrm{m}\,,\ a = 25\,\mu\,\mathrm{m}\,,\ n_1 \approx 1.4 \ \Rightarrow\ k\,a \approx 260 \ \Rightarrow\ \Delta = 0.005 \ \Rightarrow\ N \approx 335$

In contrast $a \sim 2.7\,\mu\,\mathrm{m}\,,\ \Delta \sim 0.0025,\ N \sim 2$ for each polarizatioin in single-mode

- if the indices of refraction decrease layer by layer out from the center, a ray at some angle is bent successively more toward the axis until it is totally reflected.

- for an arbitrary number of layers outside the core, the critical angle $\theta_{max}=\cos^{-1}(n_{outer}/n_{inner})$, just as for the simple two-index fiber.

- The limit of many layers is a graded index fiber in which the index of refraction varies continuously with radius from the axis.

- With the eikonal approximation, assume the medium is linear, nonconducting, nonmagnetic with an index of refraction $n(\mathbf{x})=\sqrt{\epsilon(\mathbf{x})/\epsilon_0}$ varying in space slowly

- With fields varying as $e^{-i\omega t}$, the Maxwell eqns can give Helmholtz wave eqns

$$\begin{aligned}&\nabla^2\mathbf{E}+\mu_0\omega^2\epsilon(\mathbf{x})\mathbf{E}+\cancel{\nabla(\mathbf{E}\cdot\nabla\ln\epsilon)}=0\\&\nabla^2\mathbf{H}+\mu_0\omega^2\epsilon(\mathbf{x})\mathbf{H}-\cancel{i\omega\nabla\epsilon\times\mathbf{E}}=0\end{aligned}\quad\Leftarrow\quad\epsilon\text{ changes little}\quad\Rightarrow\quad\left[\nabla^2+\frac{\omega^2}{c^2}n^2(\mathbf{x})\right]\psi=0$$

$$\Rightarrow\quad\text{plane wave}\quad\Rightarrow\quad|k(\mathbf{x})|=\frac{\omega n(\mathbf{x})}{c}\quad\Rightarrow\quad\psi=e^{i\omega S(\mathbf{x})/c}\quad\Leftarrow\quad S(\mathbf{x}):\text{eikonal}$$

$$\Rightarrow\quad\frac{\omega^2}{c^2}\left[n^2(\mathbf{x})-\nabla S\cdot\nabla S\right]+\cancel{i\frac{\omega}{c}\nabla^2 S}=0\quad\Leftarrow\quad\text{slow variation of }n+\text{small change of }S$$

$$\Rightarrow\quad\nabla S\cdot\nabla S=n^2\quad\Leftarrow\quad\textit{eikonal approximation}\text{ of quasi-geometrical optics}$$

- consider the expansion $S(\mathbf{x})=S(\mathbf{x}_0)+(\mathbf{x}-\mathbf{x}_0)\cdot\nabla S(\mathbf{x}_0)+\cdots$

$$\Rightarrow\quad\text{wave amplitude }\psi(\mathbf{x})\approx e^{i\omega\frac{S(\mathbf{x}_0)}{c}}e^{i(\mathbf{x}-\mathbf{x}_0)\cdot\frac{\omega\nabla S(\mathbf{x}_0)}{c}}$$

- The form of ϕ is that of a plane wave with wave vector

$$\mathbf{k}(\mathbf{x}_0) = \frac{\omega}{c} \nabla S(\mathbf{x}_0) = \frac{\omega}{c} n(\mathbf{x}_0)\, \hat{\mathbf{k}}(\mathbf{x}_0) \quad \Leftarrow \quad \nabla S = n(\mathbf{x})\, \hat{\mathbf{k}}(\mathbf{x})$$

- ϕ describes a wave front being locally plane & propagating in the direction of **k**.

- If we imagine advancing incrementally in the direction of **k**, we trace out a path that is the geometrical ray associated with the wave.

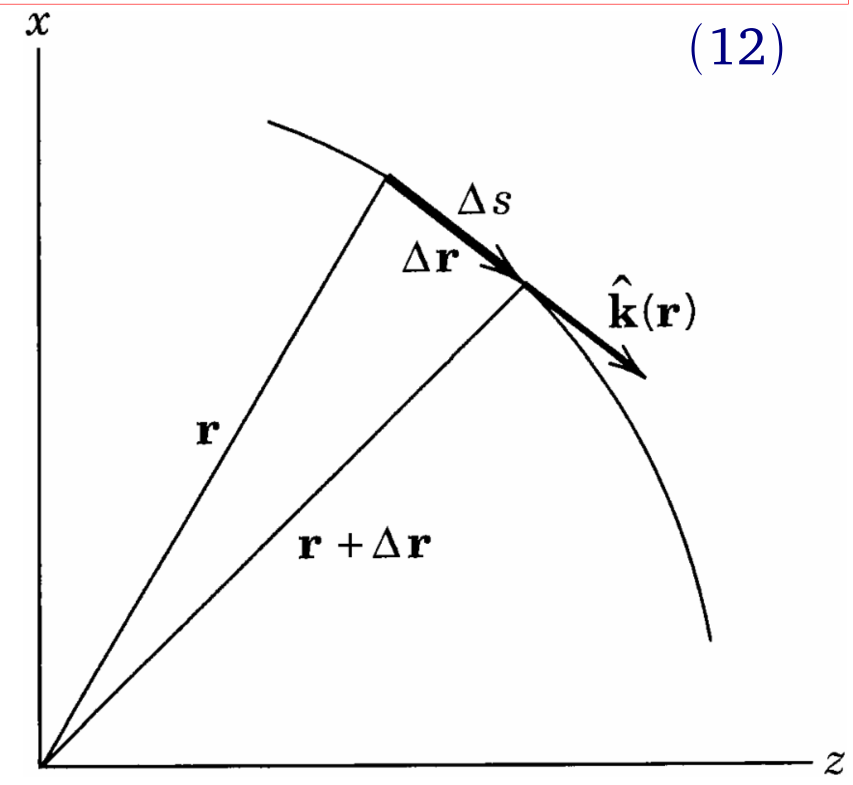
$$\hat{\mathbf{k}} \equiv \lim \frac{\delta \mathbf{r}}{\delta s} = \frac{d\mathbf{r}}{ds} \quad \Rightarrow \quad n(\mathbf{x}) \frac{d\mathbf{r}}{ds} = \nabla S \quad \Rightarrow \quad \frac{d}{ds}\left[n(\mathbf{x})\frac{d\mathbf{r}}{ds}\right] = \frac{d}{ds}\nabla S = \nabla \frac{dS}{ds}$$

$$\frac{d}{ds} = \hat{\mathbf{k}} \cdot \nabla \quad \Rightarrow \quad \frac{dS}{ds} = \hat{\mathbf{k}} \cdot \nabla S = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}\, n(\mathbf{r}) \quad \Rightarrow \quad \boxed{\frac{d}{ds}\left[n(\mathbf{r})\frac{d\mathbf{r}}{ds}\right] = \nabla n(\mathbf{r}) \quad \text{generalization of Snell's law}} \qquad (12)$$

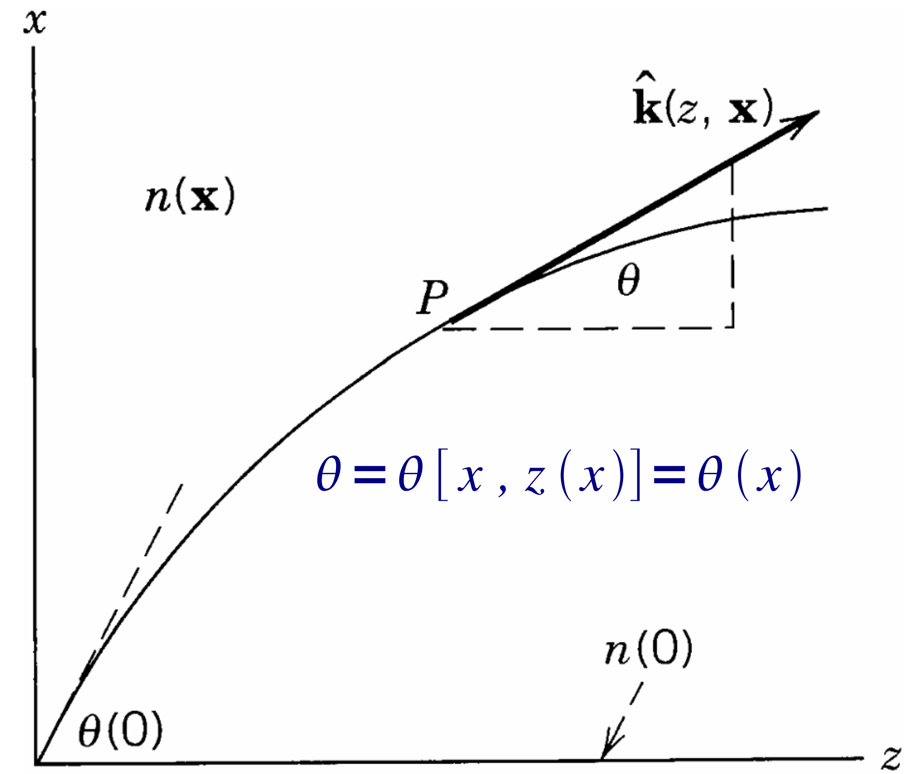
- Rays in a circular fiber fall into two classes:

1. *Meridional rays*: rays that pass through the cylinder axis; they correspond to modes with vanishing m and nonvanishing intensity at $\rho=0$.

2. *Skew rays*: rays that originate off-axis and whose path is a spiral in space with inner and outer turning points in radius; they correspond to modes with nonvanishing m and vanishing intensity at $\rho=0$.



- apply (12) only to the transmission of meridional rays in an optical fiber, or to rays in a "slab" geometry.



- $\dfrac{dx}{ds}=\sin\theta(x),\quad \dfrac{dz}{ds}=\cos\theta(x)$

$$(12)\quad\Rightarrow\quad \begin{aligned}&\frac{d}{ds}[n(x)\sin\theta(x)]=\frac{d}{dx}n(x)\\ &\frac{d}{ds}[n(x)\cos\theta(x)]=0 \qquad (13)\end{aligned}$$

$$(13)\quad\Rightarrow\quad n(x)\cos\theta(x)=n(0)\cos(0)$$

- If $n(x)$ is monotonically decreasing wrt $|x|$, there is a max (and a min) value of x attained by the ray when $\cos\theta(x_{max})=1$ $\qquad |x|=x_{max}\quad\Rightarrow\quad \bar{n}=n(x_{max})=n(0)\cos\theta(0)$

- $\bar{n}$ is a characteristic of a given ray or trajectory. From $n(x)$ we can deduce x_{max} and so delimit the lateral extent of that trajectory.

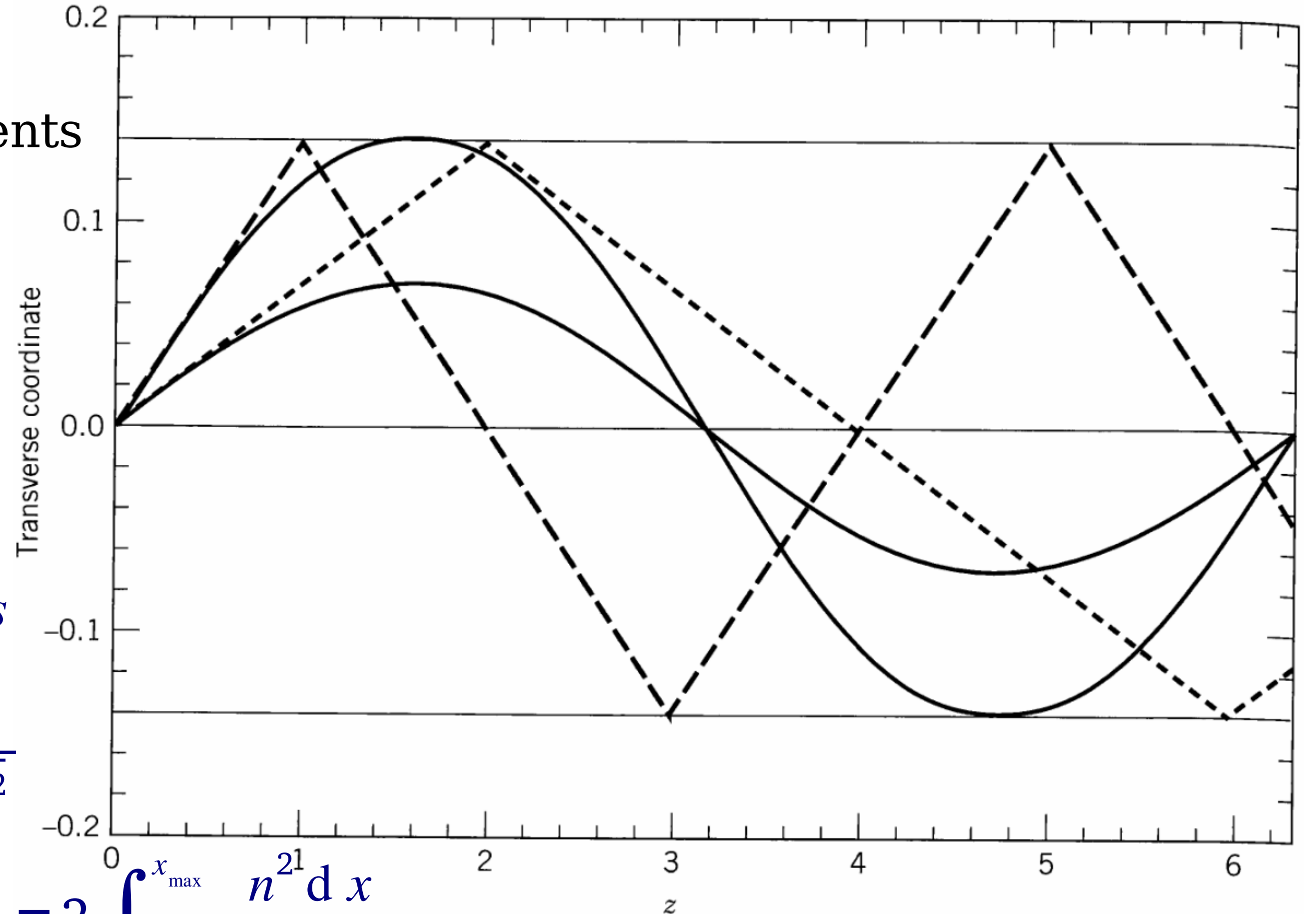
- $\bar{n}=n\cos\theta=n\dfrac{dz}{ds}\quad\Rightarrow\quad\dfrac{d}{ds}=\dfrac{\bar{n}}{n}\dfrac{d}{dz}\quad\Rightarrow\quad\dfrac{\bar{n}}{n}\dfrac{d}{dz}\left(\bar{n}\dfrac{dx}{dz}\right)=\dfrac{dn}{dx}\quad\Rightarrow\quad\bar{n}^2\dfrac{d^2x}{dz^2}=\dfrac{1}{2}\dfrac{d}{dx}n^2$

$\Rightarrow$ structure of Newton's eqn of motion $\Rightarrow$ $\bar{n}^2(dx/dz)^2=n^2-\bar{n}^2$ (energy conservation)

$$\Rightarrow\quad z(x)=\bar{n}\int_0^x\frac{dx}{\sqrt{n^2-\bar{n}^2}}\qquad\qquad \Uparrow\ \frac{dx}{dz}=0\ \text{ for }\ n(x)=\bar{n}$$

- For $x \le x_{max}$, the path represents 1/4 of a cycle of oscillation.

- $Z = 2\,\bar{n} \int_0^{x_{max}} \dfrac{\mathrm{d}\,x}{\sqrt{n^2 - \bar{n}^2}}$ half-period



- the physical & optical path lengths from A to B are

$$L_{phy} = \int_A^B \mathrm{d}\,s\,, \quad L_{opt} = \int_A^B n\,(x)\,\mathrm{d}\,s$$

$$\mathrm{d}\,s = \frac{n}{\bar{n}}\,\mathrm{d}\,z = \frac{n}{\bar{n}}\frac{\mathrm{d}\,z}{\mathrm{d}\,x}\,\mathrm{d}\,x = \frac{n\,\mathrm{d}\,x}{\sqrt{n^2 - \bar{n}^2}}$$

$$\Rightarrow \quad L_{phy} = 2\int_0^{x_{max}} \frac{n\,\mathrm{d}\,x}{\sqrt{n^2 - \bar{n}^2}}\,, \quad L_{opt} = 2\int_0^{x_{max}} \frac{n^2\,\mathrm{d}\,x}{\sqrt{n^2 - \bar{n}^2}}$$

- The transit time of a ray is $\dfrac{L_{opt}}{c} \Rightarrow T\,(z) = \dfrac{L_{opt}}{Z}\,\dfrac{z}{c}$ for $z \gg Z \Leftarrow \dfrac{c\,Z}{L_{opt}}$ group velocity

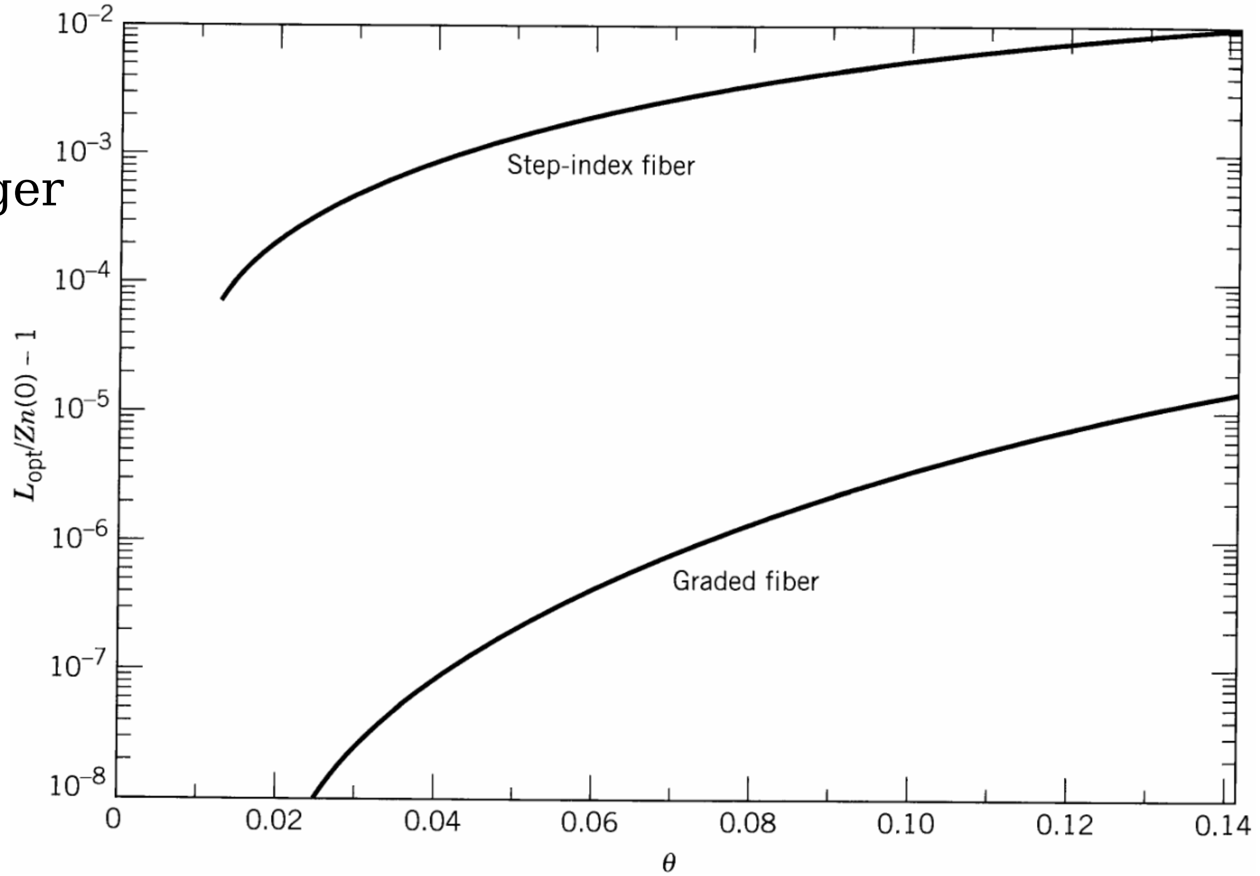
- Different rays, defined by different $\theta(0)$, have different transit times, a form of dispersion that is geometrical.

- A signal launched with a nonvanishing angular spread will be distorted unless $n(x)$ is chosen to make the transit time largely independent of n.

- rays with larger initial angles

and so larger $x_{\max}$ will have longer physical paths, but will have

larger phase velocities c/n in those longer arcs; an inherent tendency toward equalization of transit times. [problem 8.14]



8.11 Modes in Dielectric Waveguides

- the geometrical ray method of propagation is appropriate when the wavelength is short compared to the transverse dimensions of the waveguide, but the wave nature of the fields must be taken into account when the 2 scales are comparable.

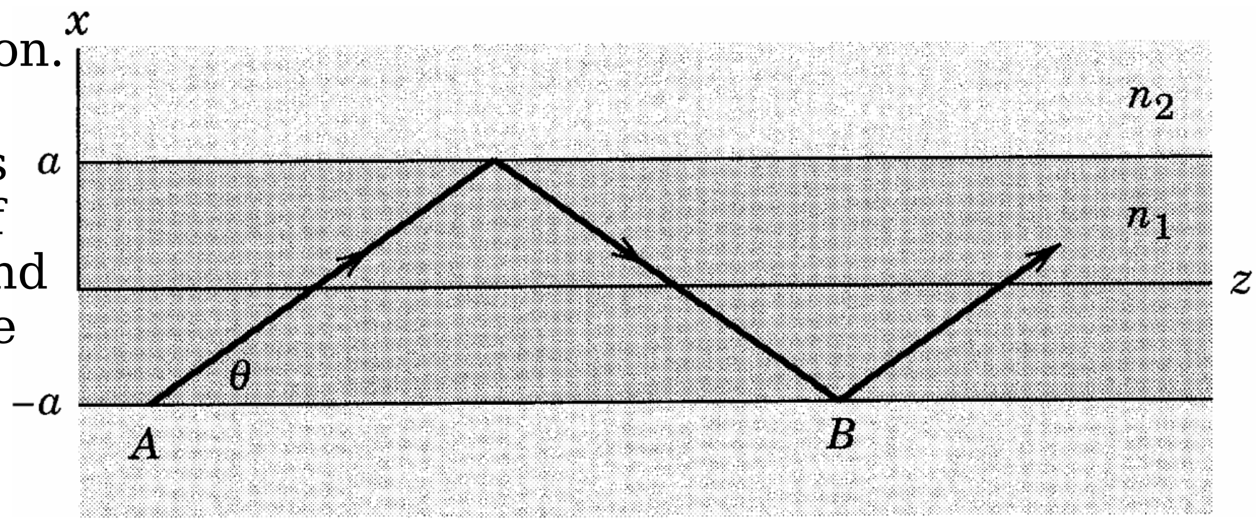
- The bound rays ($\theta<\theta_{max}$) in the geometric description are in the bound modes, with fields outside the core that decrease exponentially in the radial direction.

- Unbound rays ($\theta>\theta_{max}$) correspond to the radiating modes, with oscillatory fields outside the core.

A. Modes in a Planar Slab Dielectric Waveguide

- consider a "step-index" planar fiber consisting of a dielectric slab, any ray that makes an angle θ less than θ_{max} is totally internally reflected; the light is confined and propagates in the z direction.

- The path can be thought of as the normal to the wave front of a plane wave, reflected back and forth or alternatively as 2 plane waves, with x component of wave number, $k_x=\pm k\sin\theta$.



- To have a stable transverse field configuration and coherent propagation, the transverse phase from A to B (with 2 internal reflections) must be $2\,p\,\pi \;\Leftarrow\; p\in\mathbb{N}^{+}$

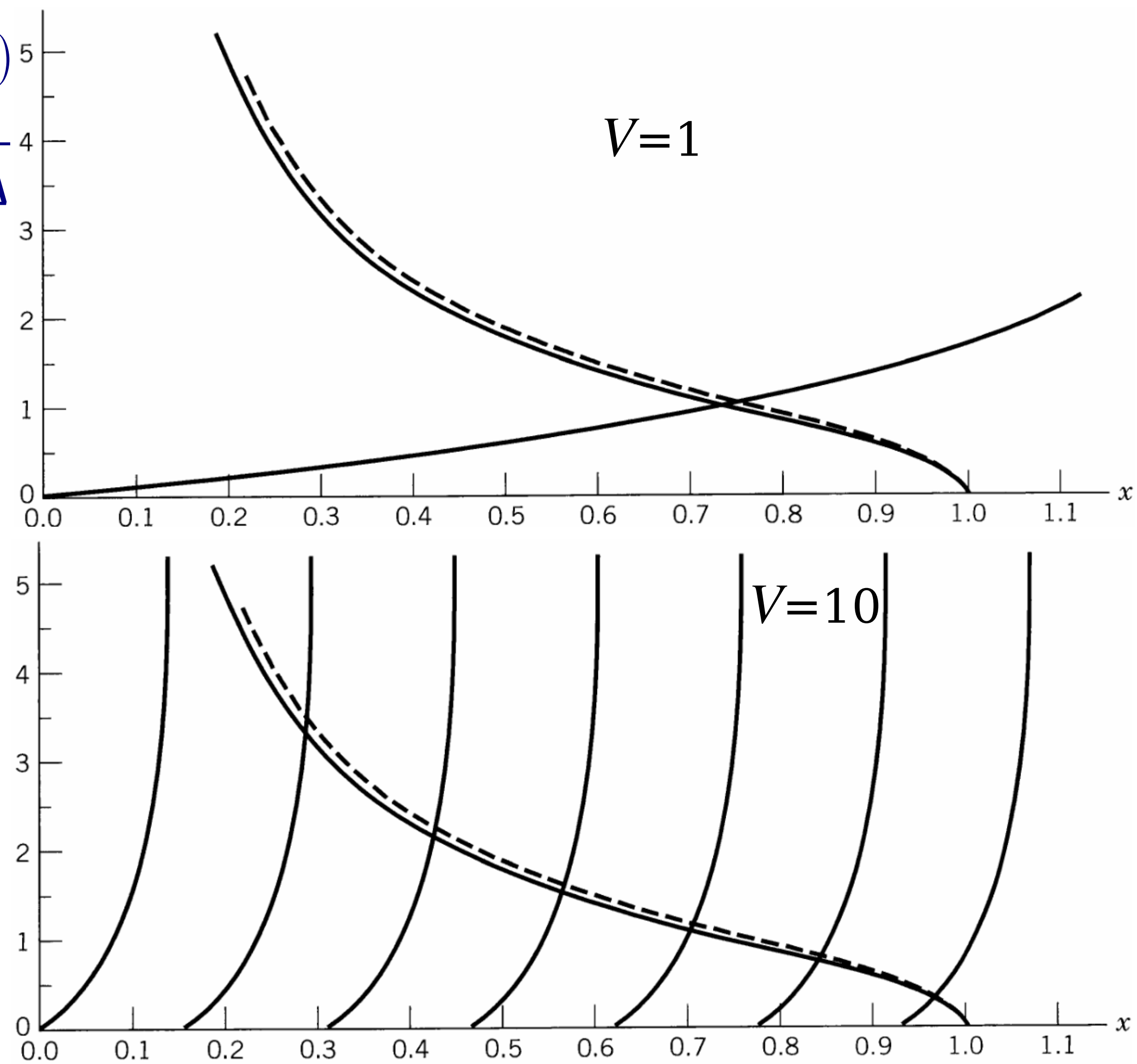
$$4\,k\,a\sin\theta+2\,\phi=2\,p\,\pi \quad\Leftarrow\quad \phi:\ \text{phase with the total internal reflection} \quad\Leftarrow\quad \begin{array}{l}\phi_{\mathrm{TE}}=-2\arctan\sqrt{2\,\Delta/\sin^2\theta-1}\\[2ex] \phi_{\mathrm{TM}}=-2\arctan\dfrac{\sqrt{2\,\Delta/\sin^2\theta-1}}{1-2\,\Delta}\end{array}$$

$$\Delta=(n_1^2-n_2^2)/2\,n_1^2 \quad\Rightarrow$$

$$\Rightarrow\quad \tan\left(V\,\xi-\frac{p\,\pi}{2}\right)=\frac{f}{\xi}\sqrt{1-\xi^2} \quad (14)$$

$$V=k\,a\sqrt{2\,\Delta}\,,\quad \xi=\sin\theta/\sqrt{2\,\Delta}$$

where $f=\begin{cases}1 & \text{for TE modes}\\ \dfrac{1}{1-2\,\Delta} & \text{for TM modes}\end{cases}$



- For small Δ the TE & TM modes are almost degenerate.

- (14) $\Rightarrow$ No. of modes $N\approx\dfrac{4\,V}{\pi}$

- Also from 1d phase-space

$$N_{\mathrm{TE}}\approx N_{\mathrm{TM}}\approx 2\,a\int_{-k_{\max}}^{k_{\max}}\frac{\mathrm{d}\,k_x}{2\,\pi}$$

$$=\frac{2\,k\,a}{\pi}\int^{\sqrt{2\,\Delta_0}}\mathrm{d}\sin\theta=\frac{2\,V}{\pi}$$

- The lowest approximation $\xi_p(\text{TE}) \approx \dfrac{\pi}{2}\dfrac{p+1}{V+1}$ $\Leftarrow$ equal spacing in p $\Leftarrow$ valid for $V \gg 1$ small p

- The fields outside the slab also affect the phase ϕ

$$(7.46) \quad \Rightarrow \quad \psi_{\text{outside}} \propto e^{-\beta|x|} \quad \Leftarrow \quad \beta = k\sqrt{2\Delta - \sin^2\theta} = \frac{V}{a}\sqrt{1-\xi^2}$$

V fixed $\Rightarrow$ β get smaller as p increases $\Rightarrow$ the fields extend farther outside

$\theta > \theta_{\max}$ $\Rightarrow$ $\xi > 1$ $\Rightarrow$ $\beta \in \Im$ $\Rightarrow$ unconfined transverse fields
The slab radiates rather than confines the fields

$\Rightarrow$ part of the power propagates within the core and part outside

$$\Rightarrow \quad P_{\text{inside}}(V=1) \simeq \frac{2}{3} P_{\text{TE}_0}, \quad P_{\text{inside}}(V \gg 1) \approx P_{\text{total}}, \quad P_{\text{outside}} \text{ is dominant for } p \approx p_{\max}$$

- $\Delta \ll 1$ $\Rightarrow$ $\theta_{\max} \approx \sqrt{2\Delta} \ll 1$ $\Rightarrow$ longitudinal propagation const $k_z = k\cos\theta \approx k$

$$\Rightarrow \quad |E_z / E_x| = \tan\theta \le \theta_{\max} = \sqrt{2\Delta} \ll 1 \text{ for the TM modes}$$

to 0^{th}-order in Δ, the TM modes have transverse electric fields and are degenerate with the TE modes.

- The combination of 2 such degenerate modes can give a mode with arbitrary direction of polarization in the x-y plane, labeled LP (for linearly polarized).

- LP modes are approximate descriptions in a circular fiber, provided $\Delta \ll 1$.

B. Modes in Circular Fibers

- For a fiber of uniform cross section with unit relative magnetic permeability and an index of refraction that does not vary along the cylinder axis but may vary in the transverse directions.

Maxwell eqn + fields $\propto e^{i(k_z z - \omega t)}$ $\Rightarrow$ $(\nabla^2 + \frac{n^2\omega^2}{c^2})\mathbf{H} = i\,\omega\,\epsilon_0 \nabla n^2 \times \mathbf{E}$

Helmholtz wave eqn $\quad \epsilon = n^2 \epsilon_0$ $\quad (\nabla^2 + \frac{n^2\omega^2}{c^2})\mathbf{E} = -\nabla(\mathbf{E}\cdot\nabla \ln n^2)$

$$(2) \Rightarrow \begin{aligned} \gamma^2 \mathbf{E}_t &= i\,[k_z \nabla_t E_z - \omega\,\mu_0 \,\hat{\mathbf{z}}\times\nabla_t H_z] \\ \gamma^2 \mathbf{H}_t &= i\,[k_z \nabla_t H_z + \omega\,\epsilon_0 n^2\, \hat{\mathbf{z}}\times\nabla_t E_z] \end{aligned} \Leftarrow \gamma^2 = \frac{n^2\omega^2}{c^2} - k_z^2 \quad \text{radial propagation constant}$$

$$\Rightarrow \begin{aligned} (\nabla_t^2 + \gamma^2) H_z - \frac{\omega^2}{\gamma^2 c^2} \nabla_t H_z \cdot \nabla_t n^2 &= -\frac{\omega\, k_z\, \epsilon_0}{\gamma^2} \hat{\mathbf{z}}\cdot\nabla_t n^2 \times \nabla_t E_z \\ (\nabla_t^2 + \gamma^2) E_z - \frac{k_z^2}{\gamma^2} \nabla_t E_z \cdot \nabla_t \ln n^2 &= \frac{\omega\, k_z\, \mu_0}{\gamma^2} \hat{\mathbf{z}}\cdot\nabla_t \ln n^2 \times \nabla_t H_z \end{aligned} \Leftarrow \text{assume } \frac{\partial n^2}{\partial z} = 0$$

- in contrast to (4), the eqns for E_z and H_z are coupled. In general there is no separation into purely TE or TM modes.

- Focus on a core of a circular cylinder of radius a with ϕ-symmetric index $n(\rho)$.

$$\Rightarrow \quad \hat{\mathbf{z}}\cdot\nabla_t n^2 \times \nabla_t \begin{bmatrix} E_z \\ H_z \end{bmatrix} = \frac{\partial n^2}{\partial \rho} \frac{1}{\rho} \frac{\partial}{\partial \phi} \begin{bmatrix} E_z \\ H_z \end{bmatrix}$$

- for a step-index fiber $\nabla_t n^2$ would vanish, at least for $\rho<a$ and for $\rho>a$; but the change from $n=n_1$ to $n=n_2$ implies a transverse gradient,

- Only if the fields have no azimuthal variation are these RHS=0; only in such circumstances are there separate TE & TM modes—$E_z=0$ or $H_z=0$.

- The modes with both E_z and H_z nonzero are known as HE or EH hybrid modes.

$$\text{continuity for } \begin{array}{c}\text{normal } \mathbf{D} \ \& \ \mathbf{B} \\ \text{tangential } \mathbf{E} \ \& \ \mathbf{H}\end{array} \text{ across } \rho = a \ + \ \text{separation of varialbes}$$

$$\Rightarrow \begin{array}{l} \begin{bmatrix} E_z \\ H_z \end{bmatrix} = \begin{bmatrix} A_e \\ A_h \end{bmatrix} J_m(\gamma\rho)\, e^{im\phi} \quad \text{for} \quad \rho < a \quad \Leftarrow \quad \gamma^2 = \dfrac{n_1^2\omega^2}{c^2} - k_z^2 \\ \begin{bmatrix} E_z \\ H_z \end{bmatrix} = \begin{bmatrix} B_e \\ B_h \end{bmatrix} K_m(\beta\rho)\, e^{im\phi} \quad \text{for} \quad \rho > a \quad \Leftarrow \quad \beta^2 = k_z^2 - \dfrac{n_2^2\omega^2}{c^2} \end{array}$$

- the TE and TM modes have nonvanishing cutoff frequencies, with the lowest for $V=n_1\omega a(2\Delta/c)^{1/2}=2.405$, the 1st root of $J_0(x)$. In contrast, the lowest HE mode (HE_{11}) has no cutoff frequency.

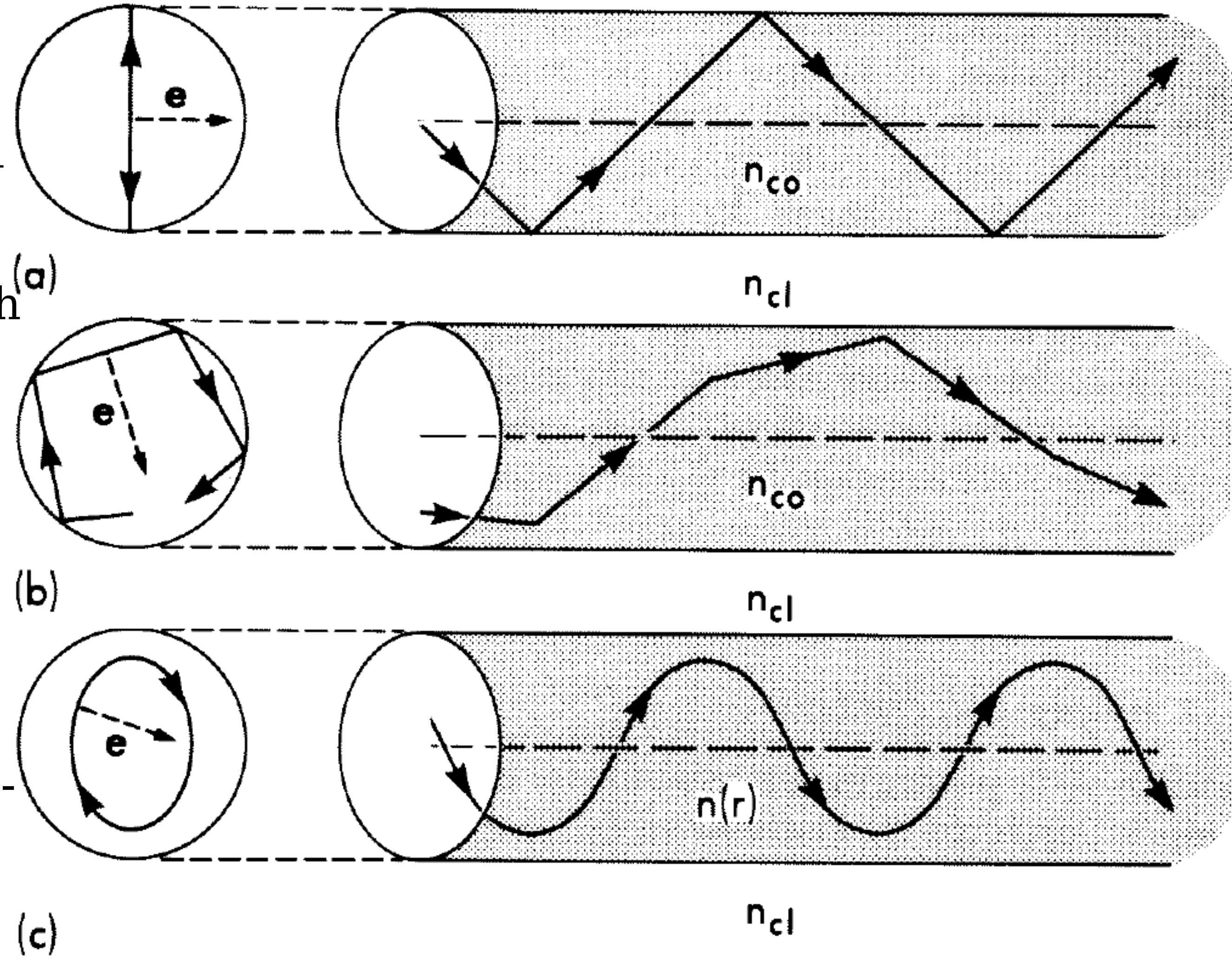
- For $0<V<2.405$, HE_{11} is the only mode that propagates in the fiber.

- The azimuthally symmetric TE or TM modes correspond to meridional rays;

- the HE or EH modes, which have azimuthal variation, correspond to skew rays.

- **E** after reflection will have a different projection on the z axis than before, as will **H**.

- Successive reflections mix TE and TM waves; the eigen-modes have both E_z and H_z Nonvanishing.



- In fibers with small Δ, called weakly guiding waveguides, the fields have small longitudinal components and are closely transverse. The language of plane light waves can be employed.

- an HE_{11} mode, with azimuthal dependence for E_z of $\cos\phi$, has fields that are approximately linearly polarized and vary as $J_0(\gamma\rho)$, labled as LP_{01}.

8.12 Expansion in Normal Modes; Fields Generated by a Localized Source in a Hollow Metallic Guide

- For any given finite frequency, only a finite number of the TE and TM modes can propagate; the rest are cutoff or evanescent modes.

- Far away from any source/obstacle/aperture in the guide, the fields are simple, with only the propagating modes (often just one) present.

- Near a source or obstacle, many modes, both propagating and evanescent, must be superposed in order to describe the fields correctly.

- The cutoff modes have sizable amplitudes only near the source or obstacle; their effects decay away over distances.

- Problems for a source/obstacle/aperture in a waveguide involves the expansion of the fields in terms of all normal modes of the guide, and a determination of the amplitudes for the propagating modes that will describe the fields far away.

A. Orthonormal Modes

- treat TE and TM modes on an equal footing.

- The fields for the λ mode propagating in the $\pm z$ direction

$$\mathbf{E}_\lambda^\pm(x,y,z)=[+\mathbf{E}_\lambda(x,y)\pm\mathbf{E}_{z\lambda}(x,y)]\,e^{\pm ik_\lambda z-i\omega t}$$
$$\mathbf{H}_\lambda^\pm(x,y,z)=[\pm\mathbf{H}_\lambda(x,y)+\mathbf{H}_{z\lambda}(x,y)]\,e^{\pm ik_\lambda z-i\omega t}$$

$\Leftarrow$ [tranverse field + longitudinal field]

$k_\lambda\in\mathbb{R}^+$ for pure propagating modes

$k_\lambda\in\mathbf{I}$ for cutoff modes

- The sign in the eqn are from the need to satisfy $\nabla\cdot\mathbf{E}=\nabla\cdot\mathbf{H}=0$ for each direction and the requirement of positive power flow in the direction of propagation.

- normalization condition by taking the transverse electric fields to be real and

$$\int_S \mathbf{E}_\lambda\cdot\mathbf{E}_\mu\,\mathrm{d}\,a=\delta_{\lambda\mu},\quad \int_S \mathbf{H}_\lambda\cdot\mathbf{H}_\mu\,\mathrm{d}\,a=\frac{\delta_{\lambda\mu}}{Z_\lambda^2},\quad \frac{1}{2}\int_S \mathbf{E}_\lambda\times\mathbf{H}_\mu\cdot\hat{\mathbf{z}}\,\mathrm{d}\,a=\frac{\delta_{\mu\lambda}}{2Z_\lambda}\quad \text{time-averaged power flow}$$

$$\Rightarrow\quad \int_S E_{z\lambda}E_{z\mu}\,\mathrm{d}\,a=-\frac{\gamma_\lambda^2}{k_\lambda^2}\delta_{\lambda\mu}\quad(\text{TM Waves}),\qquad \int_S H_{z\lambda}H_{z\mu}\,\mathrm{d}\,a=-\frac{\gamma_\lambda^2}{k_\lambda^2}\frac{\delta_{\lambda\mu}}{Z_\lambda^2}\quad(\text{TE Waves})$$

- the normalized fields in a rectangular guide for $\gamma_{mn}^2=\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$

TM

$$E_{xmn}=\frac{2\pi m}{\gamma_{mn}a\sqrt{ab}}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$E_{ymn}=\frac{2\pi n}{\gamma_{mn}b\sqrt{ab}}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b},$$

$$E_{zmn}=-i\frac{2\gamma_{mn}}{k_\lambda\sqrt{ab}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

TE

$$E_{xmn}=\frac{-2\pi n}{\gamma_{mn}b\sqrt{ab}}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$E_{ymn}=\frac{2\pi m}{\gamma_{mn}b\sqrt{ab}}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$H_{zmn}=\frac{-2i\gamma_{mn}}{k_\lambda Z_\lambda\sqrt{ab}}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$\Rightarrow\quad \mathbf{H}_t=\pm(\hat{\mathbf{z}}\times\mathbf{E}_t)/Z$$

- For TM modes, the lowest value is m=n=1. For TE modes, m=0 or n=0 is allowed. If m=0 or n=0, the normalization must be fixed by multiplying with $1/2^{1/2}$.

B. Expansion of Arbitrary Fields

- An arbitrary EM field with time dependence $e^{-i\omega t}$ can be expanded in terms of the normal mode fields

$$\mathbf{E} = \mathbf{E}^{+} + \mathbf{E}^{-} \qquad \Leftarrow \qquad \mathbf{E}^{\pm} = \sum A_{\lambda}^{\pm}\, \mathbf{E}_{\lambda}^{\pm}$$

$$\mathbf{H} = \mathbf{H}^{+} + \mathbf{H}^{-} \qquad \qquad \mathbf{H}^{\pm} = \sum A_{\lambda}^{\pm}\, \mathbf{H}_{\lambda}^{\pm}$$

- **Theorem**: The fields everywhere in the guide are determined uniquely by specification of the transverse components of **E** and **H** in a plane, z=constant.

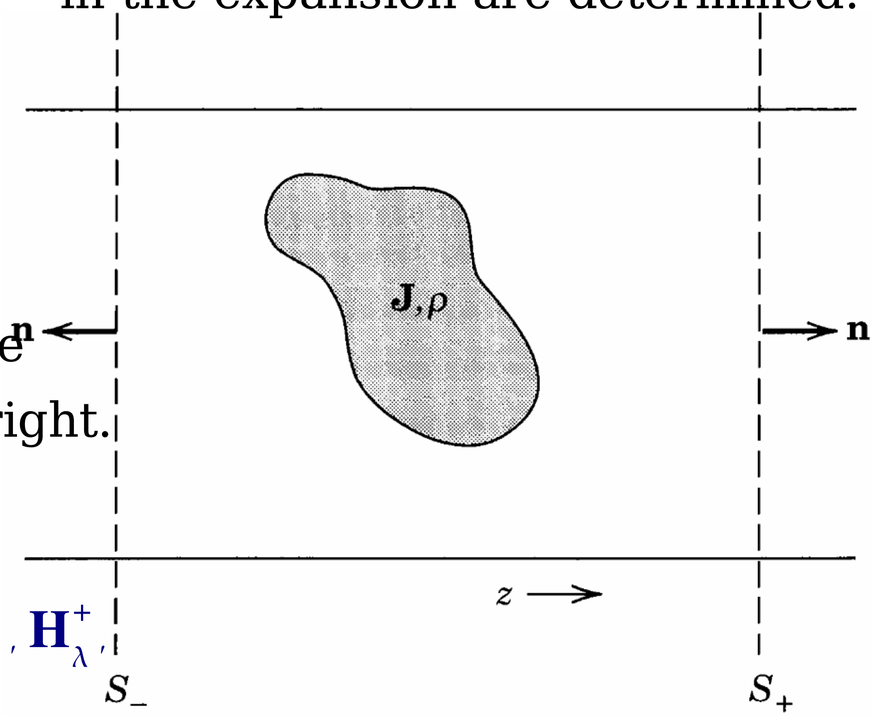
Proof: Let $z = 0\,(=\text{const}) \;\Rightarrow\; \mathbf{E}_t = \sum (A_{\lambda'}^{+} + A_{\lambda'}^{-})\,\mathbf{E}_{\lambda'}, \quad \mathbf{H}_t = \sum (A_{\lambda'}^{+} - A_{\lambda'}^{-})\,\mathbf{H}_{\lambda'} \;\Rightarrow\; \int \mathbf{E}_{\lambda}\cdot\mathbf{E}_t\, \mathrm{d}a = A_{\lambda'}^{+} + A_{\lambda'}^{-}, \quad Z_{\lambda}^{2}\int \mathbf{H}_{\lambda}\cdot\mathbf{H}_t\, \mathrm{d}a = A_{\lambda'}^{+} - A_{\lambda'}^{-}$

$$\Rightarrow \quad A_{\lambda}^{\pm} = \frac{1}{2}\int (\mathbf{E}_{\lambda}\cdot\mathbf{E}_t \pm Z_{\lambda}^{2}\,\mathbf{H}_{\lambda}\cdot\mathbf{H}_t)\,\mathrm{d}a$$

$\Rightarrow$ If $\mathbf{E}_t$ and $\mathbf{H}_t$ are given at z=0, the coefficients in the expansion are determined.

- The completeness of the normal mode expansion assures the uniqueness of the representation for all z.

C. Fields Generated by a Localized Source



- The current density is assumed to vary in time $e^{-i\omega t}$ and fields propagate to the left and to the right.

- at and to the right of the surface S_+, varying as $e^{i k_\lambda z}$ $\;\Rightarrow\; \mathbf{E} = \mathbf{E}^{+} = \sum A_{\lambda'}^{+}\,\mathbf{E}_{\lambda'}^{+}, \quad \mathbf{H} = \mathbf{H}^{+} = \sum A_{\lambda'}^{+}\,\mathbf{H}_{\lambda'}^{+}$

- On and to the left of the surface S_-, varying as

$$e^{-i k_\lambda z} \quad \Rightarrow \quad \mathbf{E} = \mathbf{E}^- = \sum A^-_{\lambda'} \mathbf{E}^-_{\lambda'}, \qquad \mathbf{H} = \mathbf{H}^- = \sum A^-_{\lambda'} \mathbf{H}^-_{\lambda'}$$

- To determine the coefficients $A_\lambda^\pm$ in terms of $\boldsymbol{J}$,

source-free Maxwell eqns for $\mathbf{E}^\pm_\lambda$, $\mathbf{H}^\pm_\lambda$ $\quad \Rightarrow \quad \nabla \cdot \left(\mathbf{E} \times \mathbf{H}^\pm_\lambda - \mathbf{E}^\pm_\lambda \times \mathbf{H} \right) = \boldsymbol{J} \cdot \mathbf{E}^\pm_\lambda$

Maxwell eqns with source for $\mathbf{E}$, $\mathbf{H}$

$$\Rightarrow \quad \int_S \left(\mathbf{E} \times \mathbf{H}^\pm_\lambda - \mathbf{E}^\pm_\lambda \times \mathbf{H} \right) \cdot \mathbf{n} \, \mathrm{d}\, a = \int_V \boldsymbol{J} \cdot \mathbf{E}^\pm_\lambda \, \mathrm{d}^3 x$$

$$= \int_{S^+} (\cdots) \cdot \hat{\mathbf{z}} \, \mathrm{d}\, a - \int_{S^-} (\cdots) \cdot \hat{\mathbf{z}} \, \mathrm{d}\, a + \cancel{\int_{\text{other}} (\cdots) \cdot \mathbf{n} \, \mathrm{d}\, a} \quad \Leftarrow$$

assume perfectly conducting walls containing no sources or apertures

$$= \hat{\mathbf{z}} \cdot \sum \left[A^+_{\lambda'} \int_{S^+} \left(\mathbf{E}^+_{\lambda'} \times \mathbf{H}^\pm_\lambda - \mathbf{E}^\pm_\lambda \times \mathbf{H}^+_{\lambda'} \right) \mathrm{d}\, a - A^-_{\lambda'} \int_{S^-} \left(\mathbf{E}^-_{\lambda'} \times \mathbf{H}^\pm_\lambda - \mathbf{E}^\pm_\lambda \times \mathbf{H}^-_{\lambda'} \right) \mathrm{d}\, a \right] = -\frac{2}{Z_\lambda} A^\mp_\lambda$$

$$\Rightarrow \quad A^\pm_\lambda = -\frac{Z_\lambda}{2} \int_V \boldsymbol{J} \cdot \mathbf{E}^\mp_\lambda \, \mathrm{d}^3 x$$

- the amplitude for propagation in the *positive* z direction comes from integration of the scalar product of the current with the mode propagating in the *negative* z direction, and vice versa.

- for the presence of apertures in the walls between the 2 planes S_+ and S_-

$$A^\pm_\lambda = \frac{Z_\lambda}{2} \int_{\text{apertures}} \mathbf{E} \times \mathbf{H}^\mp_\lambda \cdot \mathbf{n} \, \mathrm{d}\, a - \frac{Z_\lambda}{2} \int_V \boldsymbol{J} \cdot \mathbf{E}^\mp_\lambda \, \mathrm{d}^3 x \quad \Leftarrow \quad \mathbf{E}^\pm_{\lambda,\text{aperture}} \parallel \mathbf{n} \quad \Leftarrow \quad \text{Sec. 3.13}$$