

Chapter 6 Maxwell Equations, Macroscopic Electromagnetism, Conservation Laws

6.1 Maxwell's Displacement Current; Maxwell Equations

Coulomb's law $\nabla \cdot \mathbf{D} = \rho$

Ampere's law ($\nabla \cdot \mathbf{J} = 0$) $\nabla \times \mathbf{H} = \mathbf{J}$

Faraday's law $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

Absence of free magnetic poles $\nabla \cdot \mathbf{B} = 0$

- 4 equations:

- All but Faraday's law were derived from steady-state observations. Thus they are inconsistent.

- The faulty equation is Ampere's law. It is from $\nabla \cdot \mathbf{J} = 0 \leftarrow \nabla \cdot (\nabla \times \mathbf{H} = \mathbf{J}) = 0$ is valid for steady-state problems.

- the continuity equation $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \Rightarrow \nabla \cdot (\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}) = 0 \leftarrow \nabla \cdot \mathbf{D} = \rho$

$$\Rightarrow \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \leftarrow \mathbf{J} \rightarrow \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \leftarrow \frac{\partial \mathbf{D}}{\partial t} : \text{displacement current}$$

- It means that a changing electric field causes a magnetic field, even without a current. The term is of crucial importance for rapidly fluctuating fields.

- **The Maxwell equations:**

$$\nabla \cdot \mathbf{D} = \rho, \quad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

- When combined with the Lorentz force eqn and Newton's 2nd law of motion, these eqns provide a complete description of the classical dynamics of interacting charged particles and EM fields.

6.2 Vector and Scalar Potentials

- It is often convenient to introduce potentials, obtaining a smaller number of 2nd-order equations, while satisfying some of the Maxwell equations identically.

$$\nabla \cdot \mathbf{B} = 0 \Rightarrow \mathbf{B} = \nabla \times \mathbf{A} \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \Rightarrow \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{D} = \epsilon_0 \mathbf{E}, \quad \nabla \cdot \mathbf{D} = \rho \quad \Rightarrow \quad \nabla^2 \Phi + \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = -\frac{\rho}{\epsilon_0}$$

$$\mathbf{B} = \mu_0 \mathbf{H}, \quad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \Rightarrow \quad \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right) = -\mu_0 \mathbf{J}$$

Reduced 4 1st-order Maxwell equations to 2 2nd-order equations.

$$\begin{aligned} \mathbf{A} \rightarrow \mathbf{A}' &= \mathbf{A} + \nabla \Lambda \\ \Phi \rightarrow \Phi' &= \Phi - \frac{\partial \Lambda}{\partial t} \end{aligned} \Rightarrow \begin{aligned} \mathbf{B}' &= \mathbf{B} \\ \mathbf{E}' &= \mathbf{E} \end{aligned} \Rightarrow \text{choose } \boxed{\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0} \Leftarrow \begin{array}{l} \text{Lorenz} \\ \text{condition} \end{array}$$

$$\Rightarrow \boxed{\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}, \quad \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}}$$

uncouple the 2nd-order equations

6.3 Gauge Transformations, Lorenz Gauge, Coulomb Gauge

- The transformation is called a *gauge transformation*, and the invariance of the fields under such transformations is called *gauge invariance*.

- $$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \neq 0 \Rightarrow \mathbf{A}' = \mathbf{A} + \nabla \Lambda, \quad \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}$$

$$\Rightarrow \nabla \cdot \mathbf{A}' + \frac{1}{c^2} \frac{\partial \Phi'}{\partial t} = 0 = \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} + \nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2}$$

$$\Rightarrow \nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2} = - \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right) \Leftarrow \text{the new potentials } \mathbf{A}', \Phi' \text{ will satisfy the Lorenz condition and the wave eqns}$$

- $$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0 \Rightarrow \mathbf{A}' = \mathbf{A} + \nabla \Lambda, \quad \Phi' = \Phi - \frac{\partial \Lambda}{\partial t} \Rightarrow \nabla \cdot \mathbf{A}' + \frac{1}{c^2} \frac{\partial \Phi'}{\partial t} = 0$$

$$\Rightarrow \boxed{\nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2} = 0} \Leftarrow \text{restricted gauge transformation}$$

- All potentials in this restricted class are said to belong to the *Lorenz gauge*.
- The Lorenz gauge is commonly used because
 - (1) it leads to the wave eqs, which treat the potentials on equivalent footings;
 - (2) a concept indep. of the coordinate system and so fits into special relativity.

• Coulomb, radiation, or transverse gauge $\nabla \cdot \mathbf{A} = 0$ $\Rightarrow \nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$ \Leftarrow Poisson eqn

$$\Rightarrow \Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d^3x' \quad \text{instantaneous Coulomb potential} \quad (1)$$

$$\Rightarrow \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \frac{1}{c^2} \nabla \frac{\partial \Phi}{\partial t} \quad \Rightarrow \quad \nabla \times \nabla \frac{\partial \Phi}{\partial t} = 0 \quad \Leftarrow \text{irrotational}$$

$$\mathbf{J} = \mathbf{J}_\ell + \mathbf{J}_t \quad \Leftarrow \quad \begin{array}{l} \nabla \times \mathbf{J}_\ell = 0 \quad \text{longitudinal, irrotational} \\ \nabla \cdot \mathbf{J}_t = 0 \quad \text{transverse, solenoidal} \end{array} \quad + \quad \nabla \times (\nabla \times \mathbf{J}) = \nabla (\nabla \cdot \mathbf{J}) - \nabla^2 \mathbf{J}$$

$$\Rightarrow \begin{array}{l} \nabla^2 \mathbf{J}_\ell = +\nabla (\nabla \cdot \mathbf{J}) \\ \nabla^2 \mathbf{J}_t = -\nabla \times (\nabla \times \mathbf{J}) \end{array} \quad + \quad \nabla^2 \frac{1}{|\mathbf{x} - \mathbf{x}'|} = -4\pi \delta(\mathbf{x} - \mathbf{x}')$$

$$\Rightarrow \begin{array}{l} \mathbf{J}_\ell = -\frac{1}{4\pi} \nabla \int \frac{\nabla' \cdot \mathbf{J}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d^3x' \\ \mathbf{J}_t = +\frac{1}{4\pi} \nabla \times \nabla \times \int \frac{\mathbf{J}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d^3x' \end{array} \quad + \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \quad \Rightarrow \quad \frac{1}{c^2} \nabla \frac{\partial \Phi}{\partial t} = \mu_0 \mathbf{J}_\ell \quad (1)$$

$$\Rightarrow \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}_t$$

the source for the wave equation for \mathbf{A} can be expressed entirely in terms of the transverse current.

- The radiation gauge stems from that transverse radiation fields are given by the vector potential alone, the Coulomb potential contributing only to the near fields.
- This gauge is useful in quantum electrodynamics. A quantum-mechanical description of photons necessitates quantization of only the vector potential.
- The Coulomb or transverse gauge is often used with no source

$$\rho = 0, \quad \mathbf{J} = 0 \quad \Rightarrow \quad \Phi = 0, \quad \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 \quad \Rightarrow \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

- In the Coulomb gauge, the scalar potential "propagates" instantaneously, but the vector potential propagates in finite speed of propagation c .
- it is the fields, not the potentials, that matter. Another is that the transverse current extends over all space, even if \mathbf{J} is localized.

6.4 Green Functions for the Wave Equation

- Form of the wave eqn: $\nabla^2 \Psi - \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} = -4 \pi f(\mathbf{x}, t) \quad \Leftarrow \quad f : \text{source function}$

Fourier transform

$$\Psi(\mathbf{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(\mathbf{x}, \omega) e^{-i\omega t} d\omega \quad \Rightarrow \quad \Psi(\mathbf{x}, \omega) = \int_{-\infty}^{\infty} \Psi(\mathbf{x}, t) e^{i\omega t} dt$$

$$f(\mathbf{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\mathbf{x}, \omega) e^{-i\omega t} d\omega \quad \Rightarrow \quad f(\mathbf{x}, \omega) = \int_{-\infty}^{\infty} f(\mathbf{x}, t) e^{i\omega t} dt$$

$$\Rightarrow \boxed{(\nabla^2 + k^2) \Psi(\mathbf{x}, \omega) = -4 \pi f(\mathbf{x}, \omega)} \quad \Leftarrow \quad \begin{array}{l} \text{inhomogeneous Helmholtz wave equation} \\ k = \omega/c \end{array}$$

- The Helmholtz wave eqn is an elliptic partial differential eqn similar to the Poisson eqn to which it reduces for $k=0$.
- The Green function satisfies the inhomogeneous eqn

$$(\nabla^2 + k^2) G_k(\mathbf{x}, \mathbf{x}') = -4 \pi \delta(\mathbf{x} - \mathbf{x}')$$

- If there are no boundary surfaces, the Green function depend only on $\mathbf{R} = \mathbf{x} - \mathbf{x}'$ and must in fact be spherically symmetric.

$$\Rightarrow \frac{1}{R} \frac{d^2}{dR^2} (R G_k) + k^2 G_k = -4 \pi \delta(\mathbf{R}) \quad \Rightarrow \quad \frac{1}{R} \frac{d^2}{dR^2} (R G_k) + k^2 G_k = 0 \quad \text{for } R \neq 0$$

$$\Rightarrow R G_k(R) = A e^{ikR} + B e^{-ikR}$$

- the delta function has influence only at $R \rightarrow 0$. In that limit the equation reduces to the Poisson equation, since $kR \ll 1$

$$\Rightarrow \lim_{kR \rightarrow 0} G_k(R) = \frac{1}{R} \Rightarrow G_k(R) = A G_k^+(R) + B G_k^-(R) \Leftarrow G_k^\pm = \frac{e^{\pm i k R}}{R}, \quad A + B = 1$$

the 1st term represents a diverging spherical wave propagating from the origin, while the 2nd represents a converging spherical wave. The choice of A and B depends on the boundary conditions in time.

- To understand their different time behaviors, construct the corresponding time-dependent Green functions

$$\left(\nabla_x^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G^\pm(\mathbf{x}, t; \mathbf{x}', t') = -4\pi \delta(\mathbf{x} - \mathbf{x}') \delta(t - t')$$

$$\Rightarrow (\nabla_x^2 + k^2) G^\pm(\mathbf{x}, \omega; \mathbf{x}', t') = -4\pi \delta(\mathbf{x} - \mathbf{x}') e^{i\omega t'}$$

$$\Rightarrow G^\pm(R, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\pm i k R}}{R} e^{-i\omega \tau} d\omega \Leftarrow \tau \equiv t - t'$$

$$= \frac{1}{R} \delta\left(\tau \mp \frac{R}{c}\right) \text{ for } k = \frac{\omega}{c} \text{ nondispersive}$$

$$\Rightarrow G^\pm(\mathbf{x}, t; \mathbf{x}', t') = \frac{\delta\left[t' - \left(t \mp \frac{|\mathbf{x} - \mathbf{x}'|}{c}\right)\right]}{|\mathbf{x} - \mathbf{x}'|} \Leftarrow \begin{array}{l} G^+ : \text{retarded Green function} \\ G^- : \text{advanced Green function} \end{array}$$

- the retarded Green function has a causal behavior: an effect observed at the point \mathbf{x} at time t is caused by the action of a source a distance R away at an earlier or retarded time $t'=t-R/c$. Similar with the advanced Green function.

- $\Psi^\pm(\mathbf{x}, t) = \int \int G^\pm(\mathbf{x}, t; \mathbf{x}', t') f(\mathbf{x}', t') d^3 x' dt + \text{homogeneous solution}$

$$t \rightarrow -\infty \Rightarrow \left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \Psi_{\text{in}}(\mathbf{x}, t) = 0$$

$$\Rightarrow \Psi(\mathbf{x}, t) = \Psi_{\text{in}}(\mathbf{x}, t) + \int \int G^+(\mathbf{x}, t; \mathbf{x}', t') f(\mathbf{x}', t') d^3 x' dt$$

$$t \rightarrow \infty \Rightarrow \left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \Psi_{\text{out}}(\mathbf{x}, t) = 0$$

$$\Rightarrow \Psi(\mathbf{x}, t) = \Psi_{\text{out}}(\mathbf{x}, t) + \int \int G^-(\mathbf{x}, t; \mathbf{x}', t') f(\mathbf{x}', t') d^3 x' dt$$

- G^+ guarantees at remotely early times there is no contribution from the integral. G^- assures that no signal from the source exists after the source shuts off.

- usually $\Psi_{\text{in}} = 0 \Rightarrow \Psi(\mathbf{x}, t) = \int \frac{[f(\mathbf{x}', t')]_{\text{ret}}}{|\mathbf{x} - \mathbf{x}'|} d^3 x' \leftarrow \begin{array}{l} \text{evaluated at the retarded time} \\ t' = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c} \end{array}$

6.5 Retarded Solutions for the Fields: Jefimenko's Generalizations of the Coulomb and Biot-Savart Laws; Heaviside-Feynman Expressions for Fields of Point Charge

- Use the retarded solution for the wave eqns

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{[\rho(\mathbf{x}', t')]_{\text{ret}}}{R} d^3x', \quad \mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int \frac{[\mathbf{J}(\mathbf{x}', t')]_{\text{ret}}}{R} d^3x'$$

- From the 2 eqns the \mathbf{E} & \mathbf{B} fields can be computed, but it is often useful to have retarded integral solutions for the fields in terms of the sources.

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\epsilon_0} (\nabla \rho + \frac{1}{c^2} \frac{\partial \mathbf{J}}{\partial t}) \Rightarrow \mathbf{E}(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{R} \left[-\nabla' \rho - \frac{1}{c^2} \frac{\partial \mathbf{J}}{\partial t'} \right]_{\text{ret}} d^3x'$$

$$\nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = -\mu_0 \nabla \times \mathbf{J} \Rightarrow \mathbf{B}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int \frac{[\nabla' \times \mathbf{J}]_{\text{ret}}}{R} d^3x'$$

- $\nabla' [f]_{\text{ret}} \neq [\nabla' f]_{\text{ret}}$: ∇' in the retarded bracket is a spatial gradient in \mathbf{x}' with t' fixed; ∇' outside the retarded bracket is a spatial gradient in \mathbf{x}' with \mathbf{x} and t fixed.

$$[f(\mathbf{x}', t')]_{\text{ret}} = f(\mathbf{x}', t - R(t')/c) = \int f(\mathbf{x}', t') \delta[t - t' - R(t')/c] dt', \quad \hat{\mathbf{R}} = \mathbf{R}/R$$

$$\Rightarrow [\nabla' \rho]_{\text{ret}} = \nabla' [\rho]_{\text{ret}} - \left[\frac{\partial \rho}{\partial t'} \right]_{\text{ret}} \nabla' \left(t - \frac{R}{c} \right) = \nabla' [\rho]_{\text{ret}} - \left[\frac{\partial \rho}{\partial t'} \right]_{\text{ret}} \frac{\hat{\mathbf{R}}}{c}$$

$$[\nabla' \times \mathbf{J}]_{\text{ret}} = \nabla' \times [\mathbf{J}]_{\text{ret}} + \left[\frac{\partial \mathbf{J}}{\partial t'} \right]_{\text{ret}} \times \nabla' \left(t - \frac{R}{c} \right) = \nabla' \times [\mathbf{J}]_{\text{ret}} + \left[\frac{\partial \mathbf{J}}{\partial t'} \right]_{\text{ret}} \times \frac{\hat{\mathbf{R}}}{c}$$

$$\Rightarrow \mathbf{E}(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int \left[\rho(\mathbf{x}', t') \Big|_{\text{ret}} \frac{\hat{\mathbf{R}}}{R^2} + \left[\frac{\partial \rho(\mathbf{x}', t')}{\partial t'} \right]_{\text{ret}} \frac{\hat{\mathbf{R}}}{cR} - \frac{1}{c^2 R} \left[\frac{\partial \mathbf{J}(\mathbf{x}', t')}{\partial t'} \right]_{\text{ret}} \right] d^3 x'$$

$$\mathbf{B}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int \left[\mathbf{J}(\mathbf{x}', t') \Big|_{\text{ret}} \times \frac{\hat{\mathbf{R}}}{R^2} + \left[\frac{\partial \mathbf{J}(\mathbf{x}', t')}{\partial t'} \right]_{\text{ret}} \times \frac{\hat{\mathbf{R}}}{cR} \right] d^3 x'$$

Jefimenko's generalizations of the Coulomb and Biot-Savart laws

$$\Rightarrow \mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{x}') \frac{\hat{\mathbf{R}}}{R^2} d^3 x' \quad \text{for } \rho, \mathbf{J} \text{ are time-independent}$$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{x}') \times \frac{\hat{\mathbf{R}}}{R^2} d^3 x'$$

$$\left[\frac{f(\mathbf{x}', t')}{\partial t'} \right]_{\text{ret}} = \frac{\partial}{\partial t} [f(\mathbf{x}', t')]_{\text{ret}} \Leftarrow t' = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}$$

$$\rho(\mathbf{x}', t') = q \delta[\mathbf{x}' - \mathbf{r}_0(t')], \quad \mathbf{J}(\mathbf{x}', t') = \rho \mathbf{v}(t') \quad \text{for a point charge}$$

$$\Rightarrow \mathbf{E} = \frac{q}{4\pi\epsilon_0} \left[\left[\frac{\hat{\mathbf{R}}}{\kappa R^2} \right]_{\text{ret}} + \frac{1}{c} \frac{\partial}{\partial t} \left[\frac{\hat{\mathbf{R}} - \mathbf{v}/c}{\kappa R} \right]_{\text{ret}} \right] \quad (2) \quad \Leftarrow$$

$$\mathbf{B} = \frac{\mu_0 q}{4\pi} \left[\left[\frac{\mathbf{v} \times \hat{\mathbf{R}}}{\kappa R^2} \right]_{\text{ret}} + \frac{1}{c} \frac{\partial}{\partial t} \left[\frac{\mathbf{v} \times \hat{\mathbf{R}}}{\kappa R} \right]_{\text{ret}} \right]$$

Heaviside-Feynman expressions

$$\kappa = 1 - \frac{\mathbf{v} \cdot \hat{\mathbf{R}}}{c} \quad \text{retardation factor}$$

$$\delta \left[t - t' - \frac{R(t')}{c} \right] = \frac{\delta [t' - \tau(t)]}{\kappa}$$

- there is a difference between $\partial[\dots]_{\text{ret}}/\partial t$ and $[\partial\dots/\partial t]_{\text{ret}}$ because $\mathbf{x}' \rightarrow \mathbf{r}_0(t')$. The fields are functions of \mathbf{x} and t , with $t' = t - |\mathbf{x} - \mathbf{r}_0(t')|/c$.

$$\kappa = \frac{dt}{dt'} \Rightarrow \frac{1}{\kappa} = \frac{dt'}{dt} = 1 - \frac{1}{c} \frac{dR}{dt} + \mathbf{v}(t') = -\frac{dR}{dt'}$$

$$\Rightarrow (2) \Rightarrow \mathbf{E} = \frac{q}{4\pi\epsilon_0} \left[\left[\frac{\hat{\mathbf{R}}}{R^2} \right]_{\text{ret}} + \frac{[R]_{\text{ret}}}{c} \frac{\partial}{\partial t} \left[\frac{\hat{\mathbf{R}}}{R^2} \right]_{\text{ret}} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} [\hat{\mathbf{R}}]_{\text{ret}} \right] \Leftarrow \text{Feynman's expression}$$

$$\mathbf{B} = \frac{\mu_0 q}{4\pi} \left[\left[\frac{\mathbf{v} \times \hat{\mathbf{R}}}{\kappa^2 R^2} \right]_{\text{ret}} + \frac{1}{c [R]_{\text{ret}}} \frac{\partial}{\partial t} \left[\frac{\mathbf{v} \times \hat{\mathbf{R}}}{\kappa} \right]_{\text{ret}} \right] \Leftarrow \text{Heaviside's expression}$$

- The sets are equivalent.

6.6 Derivation of the Equations of Macroscopic Electromagnetism

- present a serious derivation of them from a microscopic starting point.
- The derivation remains within a classical framework.
- For dimensions large compared to 10^{-14}m , the nuclei and electrons can be treated as point systems,

$$\nabla \cdot \mathbf{b} = 0, \quad \nabla \times \mathbf{e} + \frac{\partial \mathbf{b}}{\partial t} = 0 \quad \Leftarrow \text{microscopic Maxwell equations}$$

$$\nabla \cdot \mathbf{e} = \frac{\eta}{\epsilon_0}, \quad \nabla \times \mathbf{b} - \frac{1}{c^2} \frac{\partial \mathbf{e}}{\partial t} = \mu_0 \mathbf{j}$$

no corresponding fields \mathbf{d} and \mathbf{h} because all the charges are included in η and \mathbf{j} .

- The microscopic EM fields produced by these charges vary extremely rapidly. The spatial variations occur over distances $< 10^{-10}\text{m}$, and the temporal fluctuations occur from 10^{-13}s for nuclear vibrations to 10^{-17}s for electronic orbital motion.
- All the microscopic fluctuations are averaged out, giving smooth and slowly varying macroscopic quantities, to appear in the macroscopic Maxwell equations.
- Only a spatial averaging is necessary because in a characteristic length there are still many charges, but the associated characteristic time is about the range of atomic & molecular motions, not appropriate to average out in time.

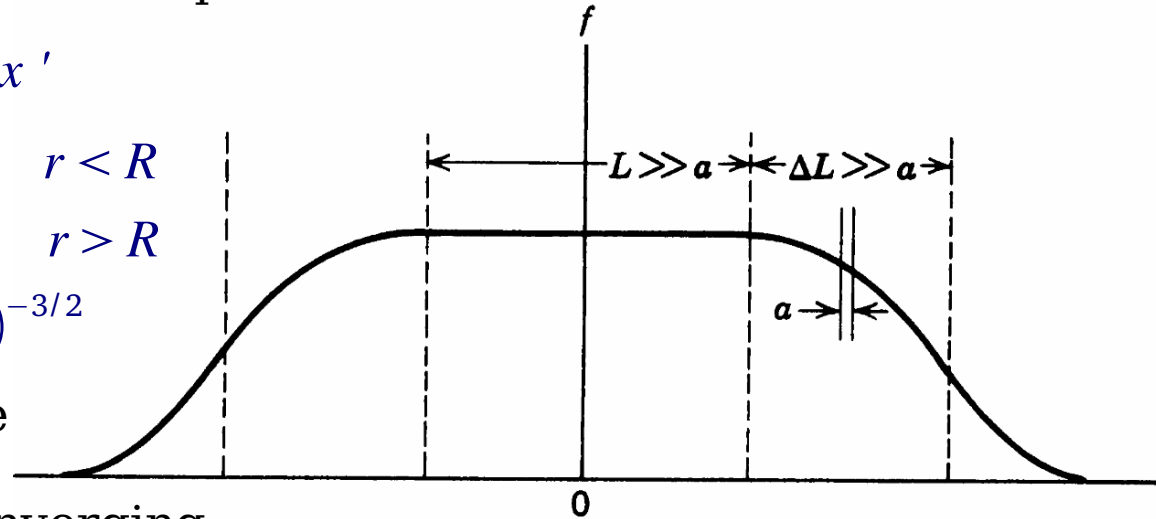
- The spatial average of a function with respect to a test function is defined as

$$\langle F(\mathbf{x}, t) \rangle = \int f(\mathbf{x}') F(\mathbf{x} - \mathbf{x}', t) d^3 x'$$

⇒ 2 examples (isotropic)

$$f(\mathbf{x}) = \begin{cases} 3/4 \pi R^3, & r < R \\ 0, & r > R \end{cases}$$

$$f(\mathbf{x}) = e^{-r^2/R^2} (\pi R^2)^{-3/2}$$



- A test function only needs to have general continuity and smoothness properties that permit a rapidly converging expansion over distances of atomic dimensions.

$$\frac{\partial}{\partial x_i} \langle F(\mathbf{x}, t) \rangle = \int f(\mathbf{x}') \frac{\partial F(\mathbf{x} - \mathbf{x}', t)}{\partial x_i} d^3 x' = \left\langle \frac{\partial F}{\partial x_i} \right\rangle, \quad \frac{\partial}{\partial t} \langle F(\mathbf{x}, t) \rangle = \left\langle \frac{\partial F}{\partial t} \right\rangle$$

The space & time differentiation commute with the averaging operation.

$$\mathbf{E}(\mathbf{x}, t) = \langle \mathbf{e}(\mathbf{x}, t) \rangle, \quad \mathbf{B}(\mathbf{x}, t) = \langle \mathbf{b}(\mathbf{x}, t) \rangle$$

$$\langle \nabla \cdot \mathbf{b} \rangle = 0 \rightarrow \nabla \cdot \mathbf{B} = 0$$

$$\epsilon_0 \nabla \cdot \mathbf{E} = \langle \eta(\mathbf{x}, t) \rangle$$

$$\Rightarrow \left\langle \nabla \times \mathbf{e} + \frac{\partial \mathbf{b}}{\partial t} \right\rangle = 0 \rightarrow \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad + \quad \frac{1}{\mu_0} \nabla \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \langle \mathbf{j}(\mathbf{x}, t) \rangle$$

homogeneous

inhomogeneous

- \mathbf{D} and \mathbf{H} are introduced by the extraction from η and \mathbf{j} of certain contributions identified with the bulk properties of the medium.

$$\eta_{\text{free}} = \sum_{j(\text{free})} q_j \delta(\mathbf{x} - \mathbf{x}_j), \quad \eta_{\text{bound}} = \sum_{n(\text{molecules})} \eta_n(\mathbf{x}, t) \Leftrightarrow \eta_n(\mathbf{x}, t) = \sum_{j(n)} q_j \delta(\mathbf{x} - \mathbf{x}_j)$$

$$\Rightarrow \eta(\mathbf{x}, t) = \sum q_j \delta[\mathbf{x} - \mathbf{x}_j(t)] = \eta_{\text{free}} + \eta_{\text{bound}}$$

$$\Rightarrow \langle \eta_n(\mathbf{x}, t) \rangle = \int f(\mathbf{x}') \eta_n(\mathbf{x} - \mathbf{x}', t) d^3 x' = \sum_{j(n)} q_j \int f(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}' - \mathbf{x}_{jn} - \mathbf{x}_n) d^3 x'$$

$$= \sum_{j(n)} q_j f(\mathbf{x} - \mathbf{x}_{jn} - \mathbf{x}_n) \Leftrightarrow \mathbf{x}_{jn} : \text{order of atom dim}$$

$$= \sum_{j(n)} q_j [f(\mathbf{x} - \mathbf{x}_n) - \mathbf{x}_{jn} \cdot \nabla f(\mathbf{x} - \mathbf{x}_n)$$

$$+ \frac{1}{2} \sum_{\alpha\beta} (\mathbf{x}_{jn})_\alpha (\mathbf{x}_{jn})_\beta \frac{\partial^2 f}{\partial x_\alpha \partial x_\beta} (\mathbf{x} - \mathbf{x}_n) + \dots]$$

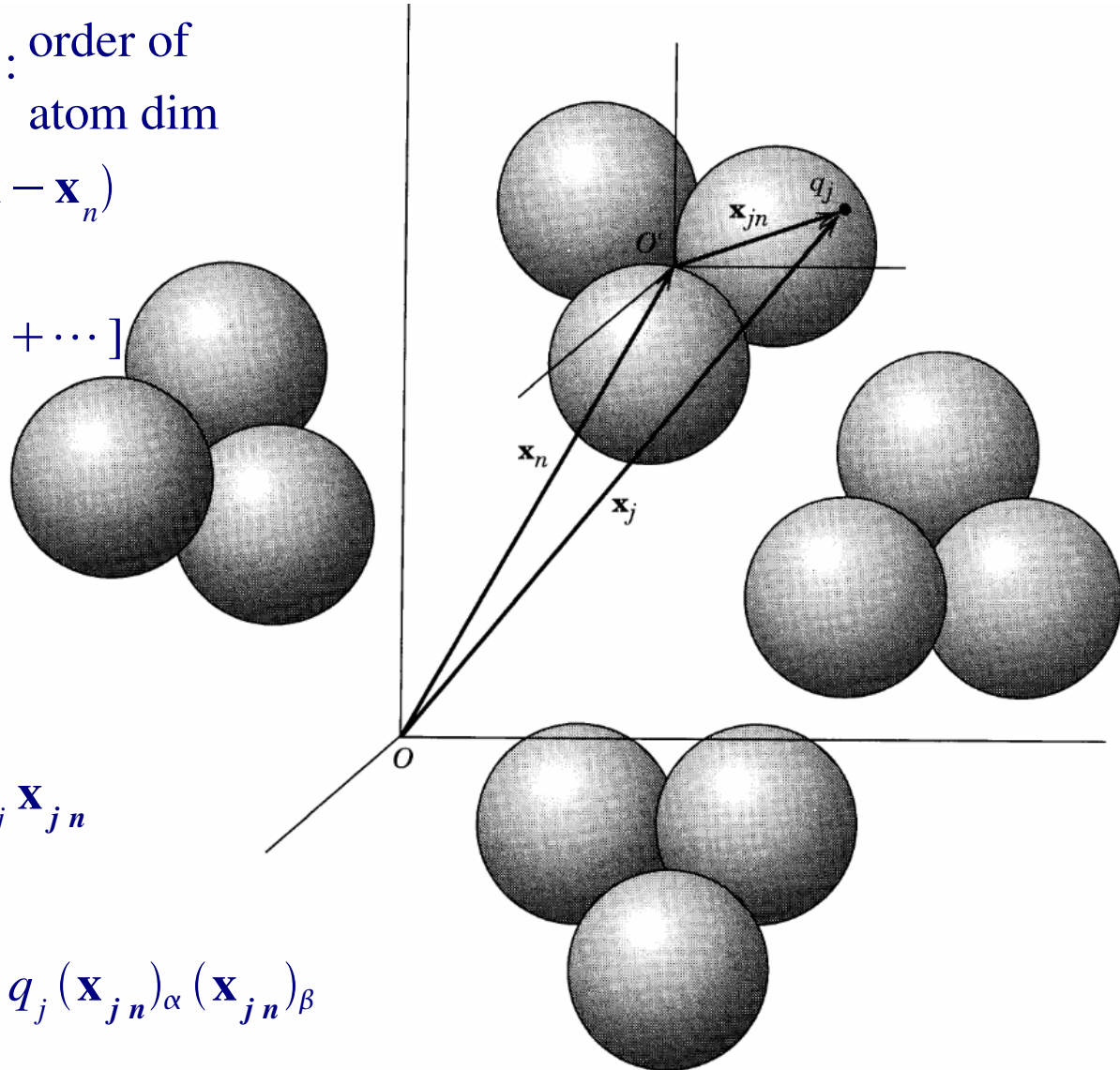
\Rightarrow *molecular multipole moments:*

molecular charge $q_n = \sum_{j(n)} q_j$

molecular dipole moment $\mathbf{p}_n = \sum_{j(n)} q_j \mathbf{x}_{jn}$

molecular quadrupole moment

$$(Q'_n)_{\alpha\beta} = 3 \sum_{j(n)} q_j (\mathbf{x}_{jn})_\alpha (\mathbf{x}_{jn})_\beta$$



$$\begin{aligned} \Rightarrow \langle \eta_n(\mathbf{x}, t) \rangle &= q_n f(\mathbf{x} - \mathbf{x}_n) - \mathbf{p}_n \cdot \nabla f(\mathbf{x} - \mathbf{x}_n) + \frac{1}{6} \sum_{\alpha\beta} (Q'_n)_{\alpha\beta} \frac{\partial^2 f}{\partial x_\alpha \partial x_\beta}(\mathbf{x} - \mathbf{x}_n) + \dots \\ &= \langle q_n \delta(\mathbf{x} - \mathbf{x}_n) \rangle - \nabla \cdot \langle \mathbf{p}_n \delta(\mathbf{x} - \mathbf{x}_n) \rangle + \frac{1}{6} \sum_{\alpha\beta} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \langle (Q'_n)_{\alpha\beta} \delta(\mathbf{x} - \mathbf{x}_n) \rangle + \dots \end{aligned}$$

- we can view the molecule as a collection of point multipoles.
- The details of the molecular charge distribution is replaced in its effect by a sum of multipoles for macroscopic phenomena.
- Fourier transforms is an alternative approach to the spatial averaging

$$g(\mathbf{x}, t) = \frac{1}{(2\pi)^3} \int \tilde{g}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}} d^3 k \quad \text{and} \quad \tilde{g}(\mathbf{k}, t) = \int g(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} d^3 x$$

$$\begin{aligned} \Rightarrow \text{convolution} & \quad f * F \equiv \int_{-\infty}^{\infty} f(y) F(x-y) dy \quad \Rightarrow \quad \mathcal{F}[f * F] = \mathcal{F}[f] \mathcal{F}[F] \\ \text{(faltung)} & \quad \mathcal{F}[f F] = \frac{\mathcal{F}[f] * \mathcal{F}[F]}{2\pi} \end{aligned}$$

$$\Rightarrow \langle F(\mathbf{x}, t) \rangle = \frac{1}{(2\pi)^3} \int \tilde{f}(\mathbf{k}) \tilde{F}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}} d^3 k \quad \Rightarrow \quad \mathcal{F}[\langle F(\mathbf{x}, t) \rangle] = \tilde{f}(\mathbf{k}) \tilde{F}(\mathbf{k}, t)$$

$$\begin{aligned} \Rightarrow \int f(\mathbf{x}) d^3 x = 1 & \quad \Rightarrow \quad \tilde{f}(0) = 1 = \int f(\mathbf{x}) e^{-i0\cdot\mathbf{x}} d^3 x \\ f = (\pi R^2)^{-3/2} e^{-r^2/R^2} & \quad \text{(Gaussian test function)} \quad \Rightarrow \quad \tilde{f}(\mathbf{k}) = \mathcal{F}[f(\mathbf{x})] = e^{-k^2 R^2/4} \end{aligned}$$

- the Fourier transform of the averaged quantity contains only low wave numbers, up to $k_{\max} = O(1/R)$, the inverse of the length scale of the averaging volume.

- $\tilde{f}(\mathbf{k} \ll 1) \rightarrow 1 \Rightarrow \mathcal{F}[F(\mathbf{x}, t)]$ gives a true representation of the long-wavelength aspects of $(F(\mathbf{x}, t))$.

- $\mathcal{F}[\langle \eta_n(\mathbf{x}, t) \rangle] = \tilde{f}(\mathbf{k}) \tilde{\eta}_n(\mathbf{k}, t) \Leftrightarrow \tilde{\eta}_n(\mathbf{k}, t) = \int \eta_n(\mathbf{x}', t) e^{-i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x}_n)} d^3 x'$

- the support for the product is confined to comparatively small wave numbers

$$\Rightarrow \tilde{\eta}_n(\mathbf{k}, t) = \tilde{\eta}_n(0, t) + \mathbf{k} \cdot \nabla_{\mathbf{k}} \tilde{\eta}_n(0, t) + \dots = \int \eta_n(\mathbf{x}', t) [1 - i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x}_n) + \dots] d^3 x'$$

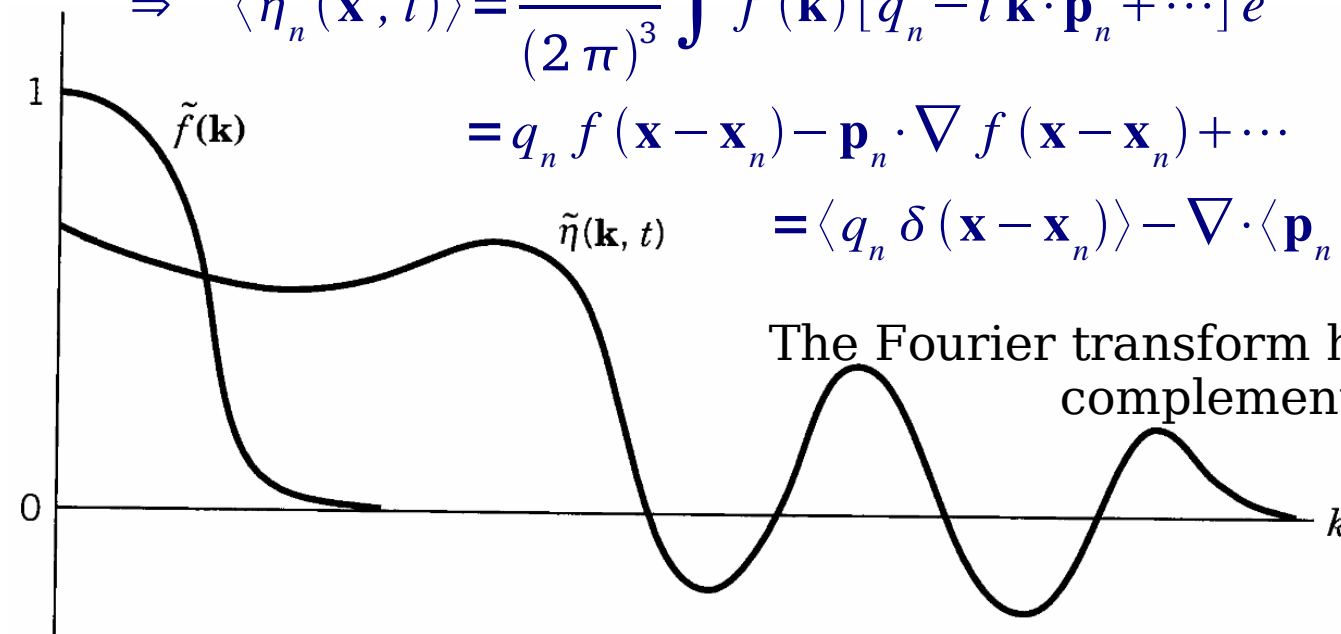
$$\Rightarrow \tilde{\eta}_n(\mathbf{k}, t) \approx q_n - i\mathbf{k} \cdot \mathbf{p}_n + \text{quadrupole \& higher}$$

$$\Rightarrow \langle \eta_n(\mathbf{x}, t) \rangle = \frac{1}{(2\pi)^3} \int \tilde{f}(\mathbf{k}) [q_n - i\mathbf{k} \cdot \mathbf{p}_n + \dots] e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}_n)} d^3 k$$

$$= q_n f(\mathbf{x} - \mathbf{x}_n) - \mathbf{p}_n \cdot \nabla f(\mathbf{x} - \mathbf{x}_n) + \dots$$

$$= \langle q_n \delta(\mathbf{x} - \mathbf{x}_n) \rangle - \nabla \cdot \langle \mathbf{p}_n \delta(\mathbf{x} - \mathbf{x}_n) \rangle + \dots$$

The Fourier transform has the advantage of giving a complementary view of the averaging as a cutoff in wave number space.



$$\Rightarrow \langle \eta(\mathbf{x}, t) \rangle = \rho(\mathbf{x}, t) - \nabla \cdot \mathbf{P}(\mathbf{x}, t) + \sum_{\alpha\beta} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} Q'_{\alpha\beta}(\mathbf{x}, t) + \dots$$

$$\rho(\mathbf{x}, t) = \left\langle \sum_{j(\text{free})} q_j \delta(\mathbf{x} - \mathbf{x}_j) + \sum_{n(\text{molecules})} q_n \delta(\mathbf{x} - \mathbf{x}_n) \right\rangle \Leftarrow \begin{array}{l} \text{macroscopic} \\ \text{charge density} \end{array}$$

$$\text{where } \mathbf{P}(\mathbf{x}, t) = \left\langle \sum_{n(\text{molecules})} \mathbf{p}_n \delta(\mathbf{x} - \mathbf{x}_n) \right\rangle \Leftarrow \begin{array}{l} \text{macroscopic} \\ \text{polarization} \end{array}$$

$$Q'_{\alpha\beta}(\mathbf{x}, t) = \frac{1}{6} \left\langle \sum_{n(\text{molecules})} (Q'_n)_{\alpha\beta} \delta(\mathbf{x} - \mathbf{x}_n) \right\rangle \Leftarrow \begin{array}{l} \text{macroscopic} \\ \text{quadrupole density} \end{array}$$

$$\Rightarrow \sum_{\alpha} \frac{\partial}{\partial x_{\alpha}} \left[\epsilon_0 E_{\alpha} + P_{\alpha} - \sum_{\beta} \frac{\partial Q'_{\alpha\beta}}{\partial x_{\beta}} + \dots \right] = \rho \Leftarrow \epsilon_0 \nabla \cdot \mathbf{E} = \langle \eta(\mathbf{x}, t) \rangle$$

$$\Rightarrow D_{\alpha} = \epsilon_0 E_{\alpha} + P_{\alpha} - \sum_{\beta} \frac{\partial Q'_{\alpha\beta}}{\partial x_{\beta}} + \dots \Leftarrow \text{macroscopic displacement vector}$$

$$\bullet \mathbf{j}(\mathbf{x}, t) = \sum q_j \mathbf{v}_j \delta(\mathbf{x} - \mathbf{x}_j(t)) = \mathbf{j}_{\text{free}} + \mathbf{j}_{\text{bound}} \Rightarrow \mathbf{j}_{\text{free}} = \sum_{j(\text{free})} q_j \mathbf{v}_j \delta(\mathbf{x} - \mathbf{x}_j)$$

$$\mathbf{j}_{\text{bound}} = \sum_{n(\text{molecules})} \mathbf{j}_n(\mathbf{x}, t) \Leftarrow \mathbf{j}_n(\mathbf{x}, t) = \sum_{j(n)} q_j \mathbf{v}_j \delta(\mathbf{x} - \mathbf{x}_j)$$

$$\Rightarrow \langle \mathbf{j}_n(\mathbf{x}, t) \rangle = \sum_{j(n)} q_j (\mathbf{v}_{j_n} + \mathbf{v}_n) f(\mathbf{x} - \mathbf{x}_n - \mathbf{x}_{j_n})$$

- The final result for the averaged microscopic current density is

$$\begin{aligned} \langle j_\alpha(\mathbf{x}, t) \rangle = & J_\alpha(\mathbf{x}, t) + \frac{\partial}{\partial t} [D_\alpha(\mathbf{x}, t) - \epsilon_0 E_\alpha(\mathbf{x}, t)] + \sum_{\beta\gamma} \epsilon_{\alpha\beta\gamma} \frac{\partial}{\partial x_\beta} M_\gamma(\mathbf{x}, t) \\ & + \sum_\beta \frac{\partial}{\partial x_\beta} \left\langle \sum_{n(\text{molecules})} [(\mathbf{p}_n)_\alpha (\mathbf{v}_n)_\beta - (\mathbf{p}_n)_\beta (\mathbf{v}_n)_\alpha] \delta(\mathbf{x} - \mathbf{x}_n) \right\rangle \\ & - \frac{1}{6} \sum_{\beta\gamma} \frac{\partial^2}{\partial x_\beta \partial x_\gamma} \left\langle \sum_{n(\text{molecules})} [(Q'_n)_{\alpha\beta} (\mathbf{v}_n)_\gamma - (Q'_n)_{\gamma\beta} (\mathbf{v}_n)_\alpha] \delta(\mathbf{x} - \mathbf{x}_n) \right\rangle + \dots \end{aligned}$$

$$\mathbf{J}(\mathbf{x}, t) = \left\langle \sum_{j(\text{free})} q_j \mathbf{v}_j \delta(\mathbf{x} - \mathbf{x}_j) + \sum_{n(\text{molecules})} q_n \mathbf{v}_n \delta(\mathbf{x} - \mathbf{x}_n) \right\rangle \Leftarrow \begin{array}{l} \text{macroscopic} \\ \text{current density} \end{array}$$

where $\mathbf{M}(\mathbf{x}, t) = \left\langle \sum_{n(\text{molecules})} \mathbf{m}_n \delta(\mathbf{x} - \mathbf{x}_n) \right\rangle \Leftarrow \begin{array}{l} \text{macroscopic} \\ \text{magnetization} \end{array}$

where $\mathbf{m}_n = \sum_{j(n)} \frac{q_j}{2} (\mathbf{x}_{jn} \times \mathbf{x}_{jn}) \Leftarrow \text{molecular magnetic moment}$

- $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}, \quad \frac{1}{\mu_0} \nabla \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \langle \mathbf{j} \rangle$

$$\Rightarrow \frac{1}{\mu_0} B_\alpha - H_\alpha = M_\alpha + \left\langle \sum_{n(\text{molecules})} (\mathbf{p}_n \times \mathbf{v}_n)_\alpha \delta(\mathbf{x} - \mathbf{x}_n) \right\rangle$$

$$- \frac{1}{6} \sum_{\beta\gamma\sigma} \epsilon_{\alpha\beta\gamma} \frac{\partial}{\partial x_\sigma} \left\langle \sum_{n(\text{molecules})} (Q'_n)_{\sigma\beta} (\mathbf{v}_n)_\gamma \delta(\mathbf{x} - \mathbf{x}_n) \right\rangle + \dots$$

- Except the 1st term of RHS, the other terms are small since \mathbf{v} 's are small.

- But if the medium as a whole has a translational velocity \mathbf{v}

$$\mathbf{v}_n = \mathbf{v} \Rightarrow \frac{1}{\mu_0} \mathbf{B} - \mathbf{H} = \mathbf{M} + (\mathbf{D} - \epsilon_0 \mathbf{E}) \times \mathbf{v}$$

- For a medium in motion the electric polarization (and quadrupole density) enter the effective magnetization.

- $Q_{\alpha\beta} \equiv \int (3 x_\alpha x_\beta - r^2 \delta_{\alpha\beta}) d^3 x$ traceless molecular quadrupole moment

$$\Rightarrow (Q'_n)_{\alpha\beta} = (Q_n)_{\alpha\beta} + e r_n^2 \delta_{\alpha\beta} \Leftarrow e r_n^2 = \sum_{j(n)} q_j x_{jn}^2$$

$$\Rightarrow Q'_{\alpha\beta} = Q_{\alpha\beta} + \frac{1}{6} \left\langle \sum_{n(\text{molecule})} e r_n^2 \delta_{\alpha\beta} \delta(\mathbf{x} - \mathbf{x}_n) \right\rangle \Leftarrow \text{macroscopic quadrupole density}$$

$$\Rightarrow \rho \rightarrow \rho_{\text{free}} + \left\langle \sum_{n(\text{molecule})} q_n \delta(\mathbf{x} - \mathbf{x}_n) \right\rangle + \frac{1}{6} \nabla^2 \left\langle \sum_{n(\text{molecule})} e r_n^2 \delta(\mathbf{x} - \mathbf{x}_n) \right\rangle \quad (**)$$

the traceless quadrupole density replaces the quadrupole density and the charge density is augmented by an additional term.

- The molecular charge & mean square radius terms represent the 1st two terms in an expansion of the $\ell=0$ molecular multipole as we go beyond the static limit.

$$F(k^2) \equiv \int \rho(\mathbf{x}) \langle e^{i\mathbf{k}\cdot\mathbf{x}} \rangle_{\ell=0 \text{ part}} d^3 x = \int \rho(\mathbf{x}) \frac{\sin k r}{k r} d^3 x \approx \int \rho d^3 x + \frac{1}{6} k^2 \int r^2 \rho d^3 x$$

$$\Rightarrow \mathbf{k} \leftrightarrow -i \nabla \Rightarrow (**)$$

6.7 Poynting's Theorem and Conservation of Energy & Momentum for a System of Charged Particles and Electromagnetic Fields

- For a single charge the rate of doing work by external EM fields is $q\mathbf{v} \cdot \mathbf{E}$. The magnetic field does no work, since the magnetic force is perpendicular to \mathbf{v} .
- the total rate of doing work by the fields in a finite volume is $\int_V \mathbf{J} \cdot \mathbf{E} d^3x$
- It represents a conversion of EM energy into mechanical or thermal energy. It must be balanced by a corresponding rate of decrease of energy in the EM field.

$$\int_V \mathbf{J} \cdot \mathbf{E} d^3x = \int_V \left[\mathbf{E} \cdot \nabla \times \mathbf{H} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \right] d^3x \quad \Leftarrow \text{Ampere-Maxwell law}$$

$$= - \int_V \left[\nabla \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right] d^3x \quad \Leftarrow \begin{array}{l} \nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H} \\ \nabla \times \mathbf{E} + \partial \mathbf{B} / \partial t = 0 \end{array}$$

$$\Rightarrow - \int_V \mathbf{J} \cdot \mathbf{E} d^3x = \int_V \left[\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{E} \times \mathbf{H}) \right] d^3x \quad \Leftarrow \begin{array}{l} (1) \text{ the medium is linear} \\ u \equiv \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) \end{array}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}} \quad (\text{Poynting theorem}) \quad \Leftarrow \quad \boxed{\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}} \quad (\text{Poynting vector})$$

- Since only its divergence appears in the conservation law, the Poynting vector seems arbitrary to the extent that the curl of any vector field can be added to it. Relativistic considerations show that it is unique.

- the time rate of change of EM energy in a volume, plus the energy flowing out through the boundary surfaces of the volume per unit time, is equal to the negative of the total work done by the fields on the sources within the volume.

- $\frac{d E_{\text{mech}}}{d t} = \int_V \mathbf{J} \cdot \mathbf{E} d^3 x$, $E_{\text{field}} = \int_V u d^3 x = \frac{\epsilon_0}{2} \int_V (E^2 + c^2 B^2) d^3 x$

$$\Rightarrow \frac{d E}{d t} = \frac{d E_{\text{mech}}}{d t} + \frac{d E_{\text{field}}}{d t} = -\oint_S \mathbf{n} \cdot \mathbf{S} d a \quad \text{Poynting's theorem}$$

- EM force on a charged particle $\mathbf{F} = q (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \Rightarrow \frac{d \mathbf{P}_{\text{mech}}}{d t} = \int_V (\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}) d^3 x$

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E}, \quad \mathbf{J} = \frac{1}{\mu_0} \nabla \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\Rightarrow \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} = \epsilon_0 \left[\mathbf{E} (\nabla \cdot \mathbf{E}) + \mathbf{B} \times \frac{\partial \mathbf{E}}{\partial t} - c^2 \mathbf{B} \times (\nabla \times \mathbf{B}) \right]$$

$$= \epsilon_0 \left[\mathbf{E} (\nabla \cdot \mathbf{E}) + c^2 \mathbf{B} (\nabla \cdot \mathbf{B}) - \mathbf{E} \times (\nabla \times \mathbf{E}) - c^2 \mathbf{B} \times (\nabla \times \mathbf{B}) - \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) \right]$$

$$\Rightarrow \frac{d \mathbf{P}_{\text{mech}}}{d t} + \frac{d \mathbf{P}_{\text{field}}}{d t} \left[\leftarrow \boxed{\mathbf{P}_{\text{field}} \equiv \epsilon_0 \int_V (\mathbf{E} \times \mathbf{B}) d^3 x = \epsilon_0 \mu_0 \int_V (\mathbf{E} \times \mathbf{H}) d^3 x} \right]$$

$$= \epsilon_0 \int_V \left[\mathbf{E} (\nabla \cdot \mathbf{E}) + c^2 \mathbf{B} (\nabla \cdot \mathbf{B}) - \mathbf{E} \times (\nabla \times \mathbf{E}) - c^2 \mathbf{B} \times (\nabla \times \mathbf{B}) \right] d^3 x$$

- the density of electromagnetic momentum $\mathbf{g} \equiv \frac{1}{c^2} \mathbf{E} \times \mathbf{H} = \epsilon_0 \mu_0 \mathbf{E} \times \mathbf{H} = \frac{1}{c^2} \mathbf{S}$

\mathbf{g} is the electromagnetic momentum associated with the fields.

- $$[\mathbf{E} \nabla \cdot \mathbf{E} - \mathbf{E} \times (\nabla \times \mathbf{E})]_1 = E_1 \left(\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3} \right) - E_2 \left(\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} \right) + E_3 \left(\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1} \right)$$

$$= \frac{\partial}{\partial x_1} E_1^2 + \frac{\partial}{\partial x_2} (E_1 E_2) + \frac{\partial}{\partial x_3} (E_1 E_3) - \frac{1}{2} \frac{\partial}{\partial x_1} (E_1^2 + E_2^2 + E_3^2)$$

$$\Rightarrow [\mathbf{E} (\nabla \cdot \mathbf{E}) - \mathbf{E} \times (\nabla \times \mathbf{E})]_\alpha = \sum_\beta \frac{\partial}{\partial x_\beta} \left(E_\alpha E_\beta - \frac{1}{2} \mathbf{E} \cdot \mathbf{E} \delta_{\alpha\beta} \right)$$

$$\Rightarrow T_{\alpha\beta} \equiv \epsilon_0 \left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2} (\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B}) \delta_{\alpha\beta} \right] \Leftarrow \text{Maxwell stress tensor}$$

$$\Rightarrow \frac{d}{dt} (\mathbf{P}_{\text{mech}} + \mathbf{P}_{\text{field}})_\alpha = \sum_\beta \int_V \frac{\partial}{\partial x_\beta} T_{\alpha\beta} d^3x = \oint_S \sum_\beta T_{\alpha\beta} n_\beta da \Leftarrow \begin{array}{l} \text{conservation of} \\ \text{linear momentum} \\ \mathbf{n} : \text{outward normal} \end{array}$$

- $\sum T_{\alpha\beta} n_\beta$ is the α^{th} component of the flow per unit area of momentum across the surface into the volume, ie, it is the force per unit area transmitted across the surface to act on the combined system of particles and fields inside the volume.

- The conservation of angular momentum of the system of particles & fields can be treated in the same way as we have handled energy & linear momentum.

6.8 Poynting's Theorem in Linear Dispersive Media with Losses

• Poynting's theorem was derived with linear media with no dispersion or losses, with ϵ and μ real and frequency independent. Actual materials exhibit dispersion and losses.

$$\begin{aligned} \mathbf{E}(\mathbf{x}, t) &= \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{x}, \omega) e^{-i\omega t} d\omega && \text{with the linearity} && \mathbf{D}(\mathbf{x}, \omega) = \epsilon(\omega) \mathbf{E}(\mathbf{x}, \omega) \\ \mathbf{D}(\mathbf{x}, t) &= \int_{-\infty}^{\infty} \mathbf{D}(\mathbf{x}, \omega) e^{-i\omega t} d\omega && && [\mathbf{B}(\mathbf{x}, \omega) = \mu(\omega) \mathbf{H}(\mathbf{x}, \omega)] \end{aligned}$$

reality of field $\Rightarrow \mathbf{E}(\mathbf{x}, -\omega) = \mathbf{E}^*(\mathbf{x}, \omega), \quad \mathbf{D}(\mathbf{x}, -\omega) = \mathbf{D}^*(\mathbf{x}, \omega), \quad \epsilon(-\omega) = \epsilon^*(\omega)$

$$\begin{aligned} \Rightarrow \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} &= \int d\omega \int \mathbf{E}^*(\omega') \cdot \mathbf{E}(\omega) [-i\omega \epsilon(\omega)] e^{-i(\omega - \omega')t} d\omega' \quad \Leftarrow \quad \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \neq \frac{\partial}{\partial t} \frac{\mathbf{E} \cdot \mathbf{D}}{2} \\ &= \frac{1}{2} \int d\omega \int \mathbf{E}^*(\omega') \cdot \mathbf{E}(\omega) [i\omega' \epsilon^*(\omega') - i\omega \epsilon(\omega)] e^{-i(\omega - \omega')t} d\omega' \quad \Leftarrow \begin{array}{l} \omega \rightarrow -\omega' \\ \omega' \rightarrow -\omega \end{array} \end{aligned}$$

assume \mathbf{E} is dominated by frequency around $\omega' \approx \omega \Rightarrow \omega' = \omega + \delta\omega \Leftarrow \delta\omega = \omega' - \omega$

$$\Rightarrow i\omega' \epsilon^*(\omega') - i\omega \epsilon(\omega) = 2\omega \Im[\epsilon(\omega)] - i(\omega - \omega') \frac{d}{d\omega} [\omega \epsilon^*(\omega)] + \dots$$

$$\begin{aligned} \Rightarrow \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} &\approx \int d\omega \int \mathbf{E}^*(\omega') \cdot \mathbf{E}(\omega) \omega \Im[\epsilon(\omega)] e^{-i(\omega - \omega')t} d\omega' \\ &+ \frac{1}{2} \frac{\partial}{\partial t} \int d\omega \int \mathbf{E}^*(\omega') \cdot \mathbf{E}(\omega) \frac{d}{d\omega} [\omega \epsilon^*(\omega)] e^{-i(\omega - \omega')t} d\omega' \quad (\#) \end{aligned}$$

$$\Rightarrow \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \approx \int d\omega \int \mathbf{H}^*(\omega') \cdot \mathbf{H}(\omega) \omega \Im[\mu(\omega)] e^{-i(\omega-\omega')t} d\omega'$$

$$+ \frac{1}{2} \frac{\partial}{\partial t} \int d\omega \int \mathbf{H}^*(\omega') \cdot \mathbf{H}(\omega) \frac{d}{d\omega} [\omega \mu^*(\omega)] e^{-i(\omega-\omega')t} d\omega' \quad (@)$$

- If ϵ and μ are real and frequency independent we go back to the last section.

- The 1st terms in (#) & (@) evidently represent the conversion of electrical energy into heat, while the 2nd terms must be an effective energy density.

- Let $\mathbf{E} = \tilde{\mathbf{E}}(t) \cos(\omega_0 t + \alpha)$, and $\tilde{\mathbf{E}}(t)$ slowly varies relative to $\frac{1}{\omega_0}$
 $\mathbf{H} = \tilde{\mathbf{H}}(t) \cos(\omega_0 t + \beta)$, $\tilde{\mathbf{H}}(t)$

$$\Rightarrow \left\langle \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right\rangle = \omega_0 \left\{ \Im[\epsilon(\omega_0)] \langle \mathbf{E}^2(\mathbf{x}, t) \rangle + \Im[\mu(\omega_0)] \langle \mathbf{H}^2(\mathbf{x}, t) \rangle \right\} + \frac{\partial u_{\text{eff}}}{\partial t}$$

$$\text{where } u_{\text{eff}} = \frac{1}{2} \Re \left[\frac{d\omega \epsilon}{d\omega}(\omega_0) \right] \langle \mathbf{E}^2(\mathbf{x}, t) \rangle + \frac{1}{2} \Re \left[\frac{d\omega \mu}{d\omega}(\omega_0) \right] \langle \mathbf{H}^2(\mathbf{x}, t) \rangle$$

- Poynting's theorem in these circumstances reads

$$\frac{\partial u_{\text{eff}}}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E} - \omega_0 \left\{ \Im[\epsilon(\omega_0)] \langle \mathbf{E}^2(\mathbf{x}, t) \rangle + \Im[\mu(\omega_0)] \langle \mathbf{H}^2(\mathbf{x}, t) \rangle \right\}$$

- The 1st term in RHS describes the explicit ohmic losses, while the next terms represent the absorptive dissipation in the medium, not counting conduction loss.

- It shows in realistic situations where, as energy flows out of the locality, there may be losses from heating of the medium, leading to a slow decay of the energy.

6.9 Poynting's Theorem for Harmonic Fields; Field Definitions of Impedance and Admittance

- assume that all fields and sources have a time dependence $e^{-i\omega t}$

$$\mathbf{E}(\mathbf{x}, t) = \Re[\mathbf{E}(\mathbf{x}) e^{-i\omega t}] \equiv \frac{1}{2} [\mathbf{E}(\mathbf{x}) e^{-i\omega t} + \mathbf{E}^*(\mathbf{x}) e^{i\omega t}]$$

$$\begin{aligned} \Rightarrow \mathbf{J}(\mathbf{x}, t) \cdot \mathbf{E}(\mathbf{x}, t) &= \frac{1}{4} [\mathbf{J}(\mathbf{x}) e^{-i\omega t} + \mathbf{J}^*(\mathbf{x}) e^{i\omega t}] \cdot [\mathbf{E}(\mathbf{x}) e^{-i\omega t} + \mathbf{E}^*(\mathbf{x}) e^{i\omega t}] \\ &= \frac{1}{2} \Re[\mathbf{J}^*(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) + \mathbf{J}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) e^{-2i\omega t}] \end{aligned}$$

- For time averages of products, the convention is to take 1/2 of the real part of the product of one complex quantity with the complex conjugate of the other.

- For harmonic fields the Maxwell equations

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0, & \nabla \times \mathbf{E} - i\omega \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{E} &= \rho, & \nabla \times \mathbf{H} + i\omega \mathbf{D} &= \mathbf{J} \end{aligned}$$

- the time-averaged rate of work done by the fields $\left\langle \frac{dW}{dt} \right\rangle = \frac{1}{2} \Re \left[\int_V \mathbf{J}^* \cdot \mathbf{E} d^3x \right]$

$$\int_V \mathbf{J}^* \cdot \mathbf{E} d^3x = \int_V \mathbf{E} \cdot [\nabla \times \mathbf{H}^* - i\omega \mathbf{D}^*] d^3x = \int_V [i\omega (\mathbf{B} \cdot \mathbf{H}^* - \mathbf{E} \cdot \mathbf{D}^*) - \nabla \cdot (\mathbf{E} \times \mathbf{H}^*)] d^3x$$

$$\mathbf{S} \equiv \frac{1}{2} (\mathbf{E} \times \mathbf{H}^*), \quad w_e = \frac{1}{4} \mathbf{E} \cdot \mathbf{D}^* \quad \text{electric energy density}, \quad w_m = \frac{1}{4} \mathbf{B} \cdot \mathbf{H}^* \quad \text{magnetic energy density}$$

$$\Rightarrow \frac{1}{2} \int_V \mathbf{J}^* \cdot \mathbf{E} d^3x + 2i\omega \int_V (w_e - w_m) d^3x + \oint_S \mathbf{S} \cdot \mathbf{n} da = 0 \quad (\text{Poynting theorem})$$

- The real part of the eqn gives the conservation of energy for the time-averaged quantities and the imaginary part relates to the reactive or stored energy and its alternating flow.

- If $\int_V (w_e - w_m) d^3 x$ is real $\Rightarrow \frac{1}{2} \int_V \Re [J^* \cdot E] d^3 x = -\oint_S \Re [S \cdot n] d a$

with lossless dielectrics and perfect conductors, the steady-state, time-averaged rate of doing work on the sources by the fields is equal to the average flow of power into the volume through the boundary surface.

- the complex power input $\frac{1}{2} I_i^* V_i = -\oint_{S_i} S \cdot n d a$

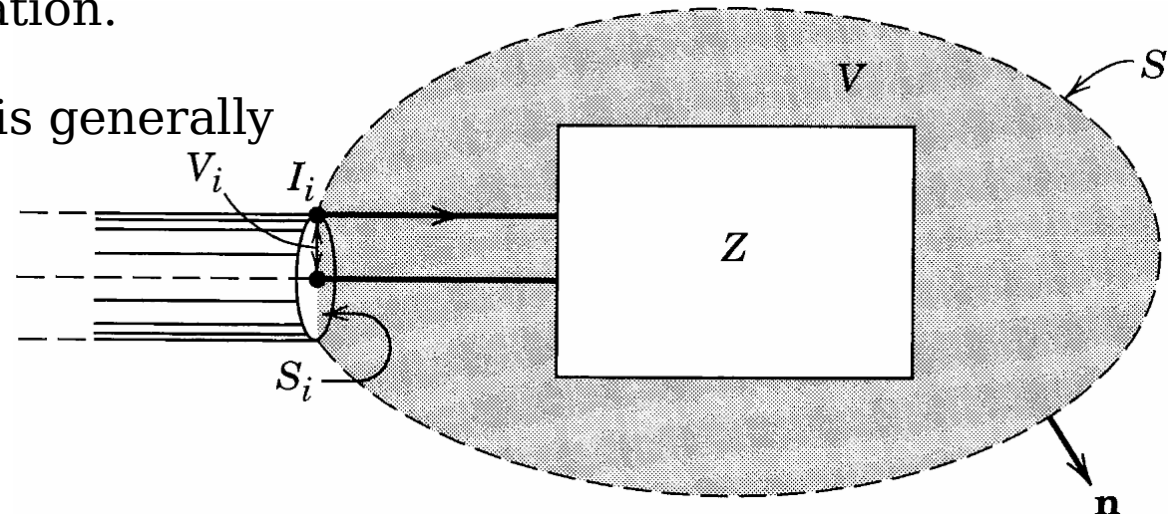
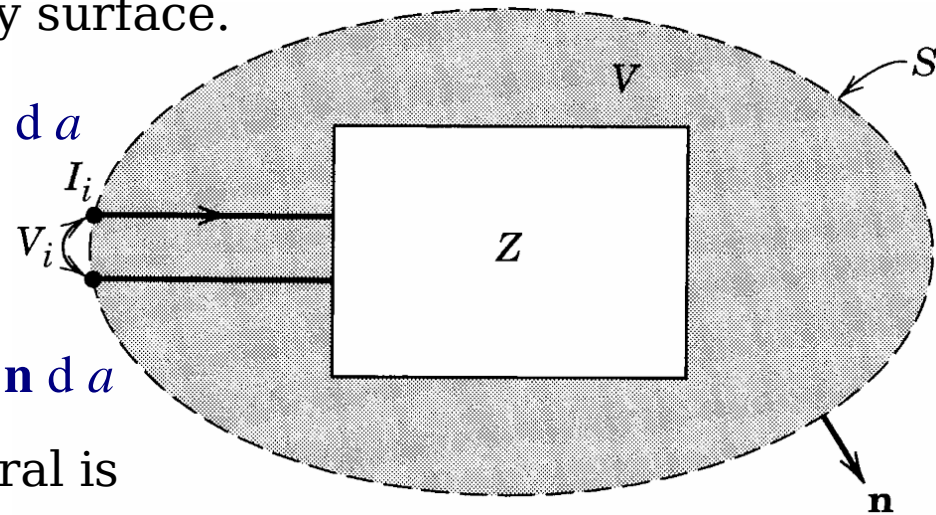
$$\Rightarrow \frac{1}{2} I_i^* V_i = \frac{1}{2} \int_V J^* \cdot E d^3 x + 2i\omega \int_V (w_e - w_m) d^3 x + \oint_{S-S_i} S \cdot n d a$$

- If $S-S_i$ is taken to infinity, the surface integral is real and represents escaping radiation.

- At low frequencies the radiation is generally negligible. No distinction need be made between S_i and S .

- The input impedance

$$Z = R - iX \Rightarrow V_i = Z I_i$$



- assume the power flow out through S is real

$$\Rightarrow R = \frac{1}{|I_i|^2} \left\{ \Re \left[\int_V \mathbf{J}^* \cdot \mathbf{E} \, d^3 x \right] + 2 \oint_{S-S_i} \mathbf{S} \cdot \mathbf{n} \, d a + 4 \omega \Im \left[\int_V (w_m - w_e) \, d^3 x \right] \right\}$$

$$X = \frac{1}{|I_i|^2} \left\{ 4 \omega \Re \left[\int_V (w_m - w_e) \, d^3 x \right] - \Im \left[\int_V \mathbf{J}^* \cdot \mathbf{E} \, d^3 x \right] \right\}$$

- $\frac{2}{I_i^2} \oint_{S-S_i} \mathbf{S} \cdot \mathbf{n} \, d a$ is the "radiation resistance," important at high frequencies.
- At low frequencies, ohmic losses are the only appreciable source of dissipation.

$$\Rightarrow R \simeq \frac{1}{|I_i|^2} \int_V \sigma |\mathbf{E}|^2 \, d^3 x, \quad X \simeq \frac{4 \omega}{|I_i|^2} \int_V (w_m - w_e) \, d^3 x$$

Here σ is the real conductivity, the energy densities are also real essentially.

- The different frequency dependences of the low-frequency reactance for inductances ($X=\omega L$) and capacitances ($X=-1/\omega C$) can be traced to the definition of L in terms of current and voltage ($V=LdI/dt$) and of C in terms of charge and voltage ($V=Q/C$).

6.10 Transformation Properties of Electromagnetic Fields and Sources Under Rotations, Spatial Reflections, and Time Reversal

A. Rotations

- A rotation in 3d is a linear transformation of the coordinates of a point and the sum of the coordinates' squares remains invariant — orthogonal transformation.

$$\bullet x'_\alpha = \sum_\beta a_{\alpha\beta} x_\beta \quad (\text{vector}) \Rightarrow \sum_\alpha a_{\alpha\beta} a_{\alpha\gamma} = \delta_{\beta\gamma} \Leftrightarrow (\mathbf{x}')^2 = \mathbf{x}^2 \Rightarrow \mathbf{a}^{-1} = \mathbf{a}^T$$

$$a^2 = 1 \Leftrightarrow a \equiv \det \mathbf{a}$$

$a = +1$: proper rotation, obtainable from the original configuration

\Rightarrow by a sequence of infinitesimal steps

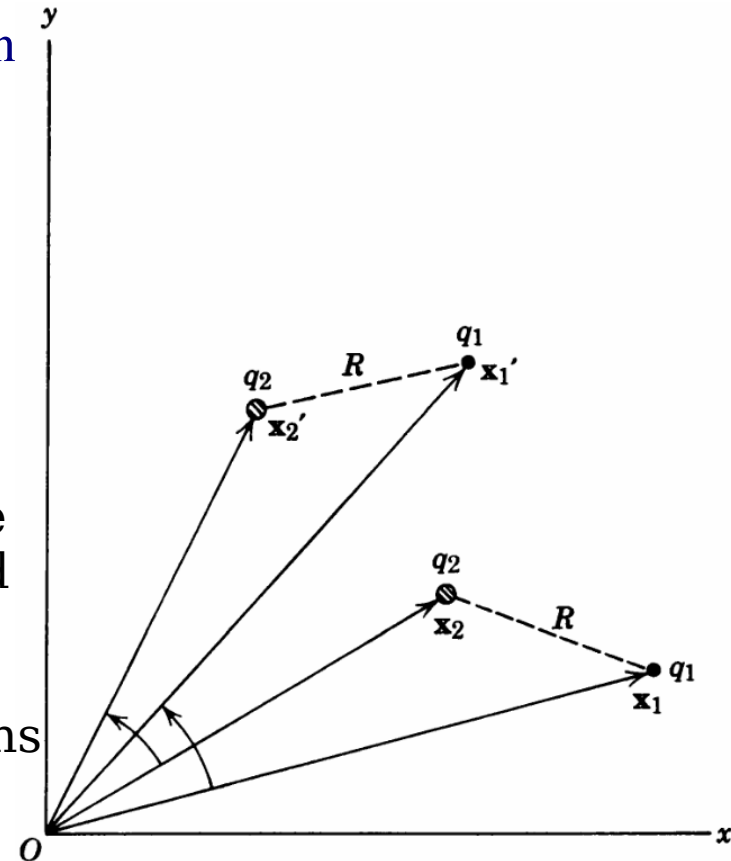
$a = -1$: improper rotation, a reflection plus a rotation

- Scalars are invariant under rotations and so are tensors of rank zero.

$$\bullet B'_{\alpha\beta} = \sum_{\gamma, \delta} a_{\alpha\gamma} a_{\beta\delta} B_{\gamma\delta} \quad \text{are called 2}^{\text{nd}}\text{-rank tensors or, commonly, tensors, like Maxwell stress tensor.}$$

- the active view of rotation—the coordinate axes are considered fixed and the physical system is imagined to undergo a rotation.

- The electrostatic potential is a scalar under rotations



- If a physical quantity is a function of coordinates and when the physical system is rotated, the quantity is unchanged, then it is a scalar function under rotations.

$$\mathbf{x} \rightarrow \mathbf{x}' \Rightarrow \phi'(\mathbf{x}') = \phi(\mathbf{x})$$

- If a set of physical quantities transform under rotation of the system according to $V'_\alpha(\mathbf{x}') = \sum_\beta a_{\alpha\beta} V_\beta(\mathbf{x})$ then they form the components of a vector.

- the gradient of a scalar transforms as a vector, the divergence of a vector is a scalar, and the Laplacian operator is a scalar.

$$\bullet \mathbf{A} = \mathbf{B} \times \mathbf{C} \Rightarrow A_\alpha = \sum_{\beta\gamma} \epsilon_{\alpha\beta\gamma} B_\beta C_\gamma \Rightarrow A'_\alpha = a \sum_\beta a_{\alpha\beta} A_\beta \Leftarrow a \equiv \det \|a_{\alpha\beta}\|$$

Under proper rotations, the cross product transforms as a vector.

B. Spatial Reflection or Inversion

- Spatial reflection in a plane is to change the signs of the \perp components of the vectors of all points and to leaving the components \parallel to the plane unchanged.

$$\text{For } x - y \text{ plane: } \mathbf{x} = (x, y, z) \rightarrow \mathbf{x}' = (x, y, -z)$$

- Space inversion corresponds to reflection of all 3 components of every vector through the origin,

$$\mathbf{x} \rightarrow \mathbf{x}' = -\mathbf{x}$$

- Spatial inversion or reflection is a discrete transformation that cannot in general be accomplished by proper rotations.

$$a = -1 \Rightarrow a_{\alpha\beta} = -\delta_{\alpha\beta} \text{ for inversion operation}$$

- *Polar vectors*: $V'_\alpha(\mathbf{x}') = \sum_\beta a_{\alpha\beta} V_\beta(\mathbf{x})$ & $\mathbf{x} \rightarrow \mathbf{x}' = -\mathbf{x} \Rightarrow \mathbf{V} \rightarrow \mathbf{V}' = -\mathbf{V}$
- *Axial or pseudovectors*: $A'_\alpha = a \sum_\beta a_{\alpha\beta} A_\beta$ & $\mathbf{x} \rightarrow \mathbf{x}' = -\mathbf{x} \Rightarrow \mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A}$
- speak of *scalars* or *pseudoscalars*, depending on whether the quantities do not or do change sign under spatial inversion.
- The triple scalar product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is a pseudoscalar if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are polar vectors.
- If a tensor of rank N transforms under spatial inversion with a factor $(-1)^N$, it is a tensor, while if the factor is $(-1)^{N+1}$ it is a pseudotensor of rank N.

C. Time Reversal

- The basic laws of physics are invariant to the sense of direction of time.
- under the time reversal transformation, the related physical quantities transform consistently so that the form of the equation is the same as before.

$$t \rightarrow t' = -t \Rightarrow \frac{d\mathbf{p}}{dt} = -\nabla U(\mathbf{x}) \text{ invariant} \Leftarrow \mathbf{x} \rightarrow \mathbf{x}' = \mathbf{x}, \mathbf{p} \rightarrow \mathbf{p}' = -\mathbf{p}$$

- if an initial configuration of a system evolves into a final configuration, a possible state of motion of the system is that the time-reversed final configuration will evolve over the reversed path to the time-reversed initial configuration.

Physical Quantity	Symbol	Rotation (rank of tensor)	Space Inversion (name)	Time Reversal
Coordinate	\mathbf{x}	1	Odd(vector)	Even
Velocity	\mathbf{v}	1	Odd(vector)	Odd
Momentum	\mathbf{p}	1	Odd(vector)	Odd
Angular Momentum	$\mathbf{L} = \mathbf{x} \times \mathbf{p}$	1	Even(pseudovector)	Odd
Force	\mathbf{F}	1	Odd(vector)	Even
Torque	$\mathbf{N} = \mathbf{x} \times \mathbf{F}$	1	Even(pseudovector)	Even
Kinetic energy	$p^2 / 2m$	0	Even(scalar)	Even
Potential energy	$U(\mathbf{x})$	0	Even(scalar)	Even
Charge density	ρ	0	Even(scalar)	Even
Current density	\mathbf{J}	1	Odd(vector)	Odd
Electric field	\mathbf{E}	1	Odd(vector)	Even
Polarization	\mathbf{P}			
Displacement	\mathbf{D}			
Magnetic induction	\mathbf{B}			
Magnetization	\mathbf{M}			
Magnetic field	\mathbf{H}	1	Even(pseudovector)	Odd
Poynting vector	$\mathbf{S} = \mathbf{E} \times \mathbf{H}$	1	Odd(vector)	Odd
Maxwell stress tensor	$T_{\alpha\beta}$	2	Even(tensor)	Even

D. Electromagnetic Quantities

- the forms of the eqns governing EM phenomena are invariant under rotations, space inversion, and time reversal.
- electric charge is invariant under Galilean & Lorentz transformations and is a scalar under rotations. Thus people assume that charge is also a scalar under spatial inversion & time reversal.
- The transformation properties of fields like \mathbf{E} & \mathbf{B} depend on the convention chosen for the charge.
- charge density is a scalar. And \mathbf{E} is a polar vector, even under time reversal.
- $\nabla \times \mathbf{E}$: pseudovector, even under time reversal $\Leftarrow \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$ (Faraday's law)
 - $\Rightarrow \mathbf{B}$: pseudovector, odd under time reversal
- the Ampere-Maxwell eqn $\frac{1}{\mu_0} \nabla \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J}$ polar vector, odd under time reversal on both sides.
- consider the structure of a constitutive relation specifying the polarization \mathbf{P} , assuming 1st order in \mathbf{E} ,
 - $\mathbf{B}_0 : \mathbf{E} \times \mathbf{B}_0, \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}_0, \frac{\partial^2 \mathbf{E}}{\partial t^2} \times \mathbf{B}_0, \dots$ odd time derivatives of \mathbf{E}
 - $\mathbf{B}_0^2 : (\mathbf{B}_0 \cdot \mathbf{B}_0) \mathbf{E}, (\mathbf{E} \cdot \mathbf{B}_0) \mathbf{B}_0, (\mathbf{B}_0 \cdot \mathbf{B}_0) \frac{\partial \mathbf{E}}{\partial t}, \dots$ zero or even time derivatives of \mathbf{E}

- Up to 2nd-order of $\mathbf{B}_0 \Rightarrow \frac{1}{\epsilon_0} \mathbf{P} = \chi_0 \mathbf{E} + \chi_1 \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}_0 + \chi_2 (\mathbf{B}_0 \cdot \mathbf{B}_0) \mathbf{E} + \chi_3 (\mathbf{E} \cdot \mathbf{B}_0) \mathbf{B}_0 + \dots$

odd time derivatives of \mathbf{E} for the terms linear in \mathbf{B}_0

even time derivatives of \mathbf{E} for the 0th and 2nd powers of \mathbf{B}_0 .

- At low frequencies, the response of all material systems is via electric force

$$\Rightarrow \frac{1}{\epsilon_0} \mathbf{P} = \chi_0 \mathbf{E} + \chi_1 \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}_0 + \chi'_2 (\mathbf{B}_0 \cdot \mathbf{B}_0) \frac{\partial^2 \mathbf{E}}{\partial t^2} + \chi'_3 \left(\frac{\partial^2 \mathbf{E}}{\partial t^2} \cdot \mathbf{B}_0 \right) \mathbf{B}_0 + \dots \text{ more realistic}$$

- At optical frequencies this equation permits an understanding of the gyrotropic behavior of waves in an isotropic medium in a constant magnetic field.

- In certain circumstances the constraints of space-time symmetries must be relaxed in constitutive relations. The optical rotatory power of chiral molecules

has the constitutive relations, $\mathbf{P} = \epsilon_0 \chi_0 \mathbf{E} + \xi \frac{\partial \mathbf{B}}{\partial t}$ and $\mu_0 \mathbf{M} = \chi'_0 \mathbf{B} + \xi' \frac{\partial \mathbf{E}}{\partial t}$

The terms involve pseudoscalar quantities ξ and ξ' that reflect the underlying lack of parity symmetry for chiral substances.

6.11 On the Question of Magnetic Monopoles

- No experimental evidence for the existence of magnetic charges or monopoles.
- Dirac's argument is that the existence of magnetic monopole would offer an explanation of the discrete nature of electric charge.
- If there exist magnetic charge & current densities, then Maxwell eqns become

$$\begin{aligned} \nabla \cdot \mathbf{D} = \rho_e, \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}_e &\Leftrightarrow \frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{J}_e = 0 \\ \nabla \cdot \mathbf{B} = \rho_m, \quad -\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} + \mathbf{J}_m &\Leftrightarrow \frac{\partial \rho_m}{\partial t} + \nabla \cdot \mathbf{J}_m = 0 \end{aligned}$$

- duality transformation: $\mathbf{E} = +\mathbf{E}' \cos \xi + Z_0 \mathbf{H}' \sin \xi$, $Z_0 \mathbf{D} = +Z_0 \mathbf{D}' \cos \xi + \mathbf{B}' \sin \xi$
impedance $Z_0 = \sqrt{\mu_0 / \epsilon_0} \Rightarrow Z_0 \mathbf{H} = -\mathbf{E}' \sin \xi + Z_0 \mathbf{H}' \cos \xi$, $\mathbf{B} = -Z_0 \mathbf{D}' \sin \xi + \mathbf{B}' \cos \xi$
For a real (pseudoscalar) angle ξ , the transformation leaves quadratic forms like $\mathbf{E} \times \mathbf{H}$, $(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$, and the components of the Maxwell stress tensor invariant.

- Apply to the sources
 $Z_0 \rho_e = +Z_0 \rho'_e \cos \xi + \rho'_m \sin \xi$, $Z_0 \mathbf{J}_e = +Z_0 \mathbf{J}'_e \cos \xi + \mathbf{J}'_m \sin \xi$
 $\rho_m = -Z_0 \rho'_e \sin \xi + \rho'_m \cos \xi$, $\mathbf{J}_m = -Z_0 \mathbf{J}'_e \sin \xi + \mathbf{J}'_m \cos \xi$
 \Rightarrow the generalized Maxwell equations are invariant

$$\nabla \cdot \mathbf{D}' = \rho'_e, \quad \nabla \times \mathbf{H}' = \frac{\partial \mathbf{D}'}{\partial t} + \mathbf{J}'_e, \quad \nabla \cdot \mathbf{B}' = \rho'_m, \quad -\nabla \times \mathbf{E}' = \frac{\partial \mathbf{B}'}{\partial t} + \mathbf{J}'_m$$

- The invariance of the EM eqns under duality transformations shows that it is only a convention to think of electric charge instead of magnetic charge.
- If *all* particles have the same ratio of magnetic to electric charge, one can make a duality transformation so that $\rho_m=0$, $\mathbf{J}_m=0$, so we have the usual Maxwell eqns.
- If we choose the electric and magnetic charges of an electron $q_e=-e$, $q_m=0$, it is known that for a proton, $q_e=+e$ and $|q_m(\text{nucleon})| < 2 \times 10^{-24} Z_0 e$.
- The conclusion is that the particles of ordinary matter possess only electric charge or they all have the same ratio of magnetic to electric charge.
- ρ_m is a pseudoscalar density, odd under time reversal, and \mathbf{J}_m is a pseudovector density, even under time reversal.
- Since the symmetries of ρ_m under spatial inversion & time reversal are opposite to those of ρ_e , then a particle with both electric & magnetic charges that space inversion and time reversal are no longer valid symmetries of the laws of physics.
- the present evidence is that the symmetry violation is extremely small and associated with the weak interactions.

- Considering the quantum mechanics of an electron near a magnetic monopole, Dirac showed that consistency required the quantization condition,

$$\frac{e g}{4 \pi \hbar} = \frac{\alpha g}{Z_0 e} = \frac{n}{2} \quad n = 0, \pm 1, \pm 2, \dots \quad \Leftarrow \quad \begin{array}{l} g : \text{magnetic charge} \\ \alpha = \frac{e^2}{4 \pi \epsilon_0 \hbar c} \approx \frac{1}{137} \quad (\text{fine structure const}) \end{array}$$

- The discrete nature of electric charge thus follows from the existence of a monopole. The magnitude of e is determined in terms of the magnetic charge g .
- With the known value of the fine structure constant, we can also infer the existence of magnetic monopoles with the magnetic "fine structure" constant

$$\frac{g^2}{4 \pi \mu_0 \hbar c} = \frac{n^2}{4} \frac{4 \pi \epsilon_0 \hbar c}{e^2} = \frac{137}{4} n^2 \quad \Leftarrow \quad \text{Dirac monopoles}$$

- The coupling strength is enormous, making their extraction from matter with dc magnetic fields and their subsequent detection very simple in principle.

6.12 Discussion of the Dirac Quantization Condition

- We consider the deflection at large impact parameters of a particle by the field of a stationary magnetic monopole.

- For $b \gg 0$, the change in the state of motion of the charged particle can be determined by the impulse of the force, assuming the particle is undeflected.

- $$\mathbf{B} = \frac{g}{4\pi r^3} \mathbf{r} \Rightarrow F_x \approx 0$$

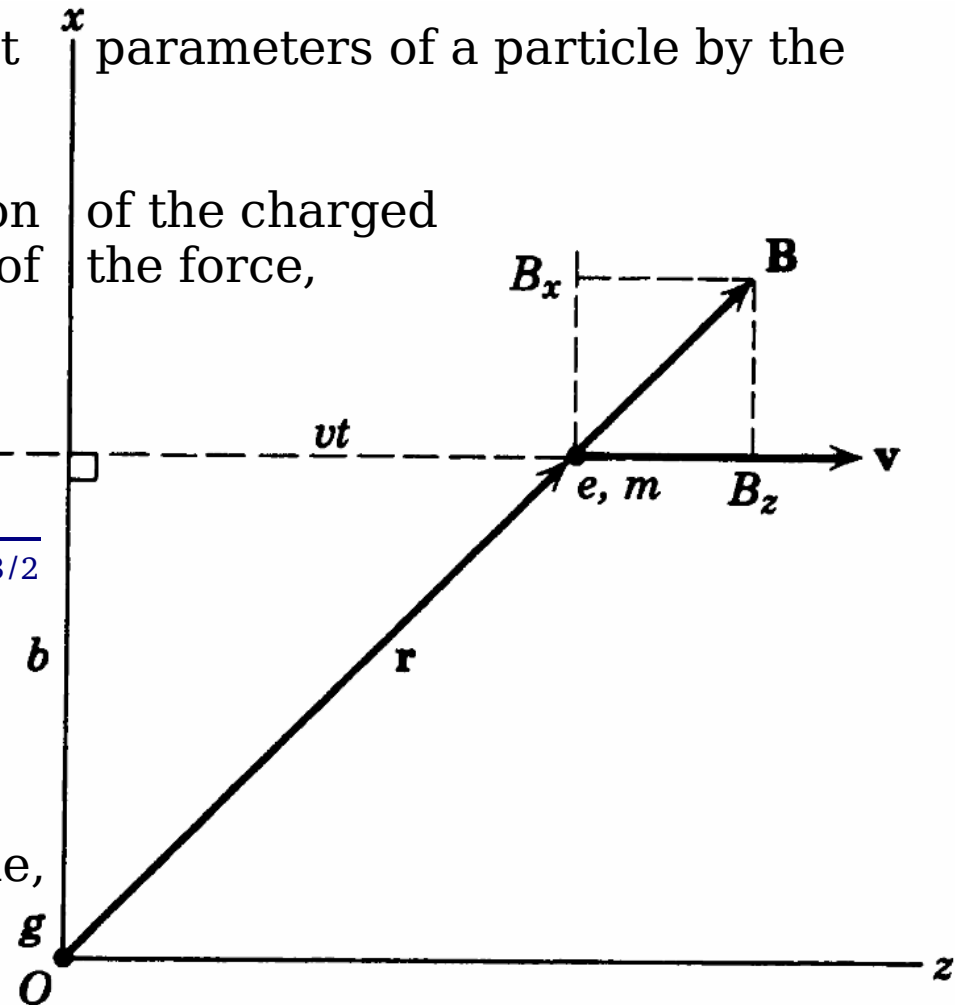
$$F_y = e v B_x = \frac{e g}{4\pi} \frac{v b}{(b^2 + v^2 t^2)^{3/2}}$$

$$\Rightarrow \Delta p_y = \frac{e g v b}{4\pi} \int_{-\infty}^{\infty} \frac{dt}{(b^2 + v^2 t^2)^{3/2}} = \frac{e g}{2\pi b}$$

- the particle is deflected out of the plane, ie, in the azimuthal direction. So the particle's angular momentum is changed

$$\Delta L_z = b \Delta p_y = \frac{e g}{2\pi}$$

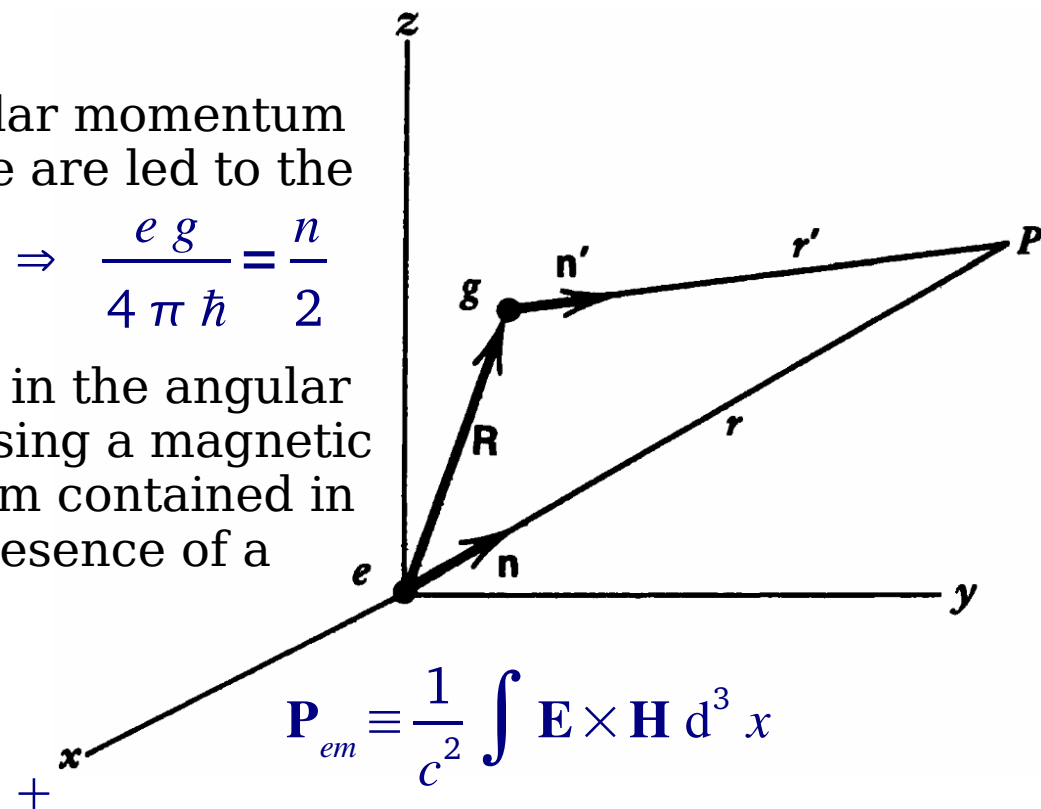
- The change in the z component of angular momentum of the particle depends only on the product eg and is a universal value for a charged particle passing a stationary monopole, no matter how far away.



- If we assume that any change of angular momentum must occur in integral multiples of \hbar , we are led to the Dirac quantization condition.

$$\frac{eg}{2\pi} = n\hbar \Rightarrow \frac{eg}{4\pi\hbar} = \frac{n}{2}$$

- The universal character of the change in the angular momentum of a charged particle in passing a magnetic monopole is from the angular momentum contained in the fields of an electric charge in the presence of a magnetic monopole.



- $\mathbf{H} = -\frac{g}{4\pi\mu_0} \nabla \frac{1}{r'} = \frac{g}{4\pi\mu_0} \frac{\mathbf{n}'}{r'^2}$

$$\mathbf{P}_{em} \equiv \frac{1}{c^2} \int \mathbf{E} \times \mathbf{H} d^3x$$

$$\mathbf{E} = -\frac{e}{4\pi\epsilon_0} \nabla \frac{1}{r} = \frac{e}{4\pi\epsilon_0} \frac{\mathbf{n}}{r^2}$$

$$\mathbf{L}_{em} \equiv \frac{1}{c^2} \int \mathbf{x} \times (\mathbf{E} \times \mathbf{H}) d^3x$$

$$\Rightarrow \mathbf{P}_{em} = 0 \quad \Leftarrow \quad \text{by symmetry} \quad \Rightarrow \quad \mathbf{L}_{em} \text{ is independent of choice of origin}$$

$$\Rightarrow \frac{4\pi}{\mu_0} \mathbf{L}_{em} = e \int \frac{1}{r} \mathbf{n} \times (\mathbf{n} \times \mathbf{H}) d^3x = e \int \frac{1}{r} [\mathbf{n}(\mathbf{n} \cdot \mathbf{H}) - \mathbf{H}] d^3x$$

$$\begin{aligned} \Rightarrow 4\pi \mathbf{L}_{em} &= -e \int (\mathbf{B} \cdot \nabla) \mathbf{n} d^3x \quad \Leftarrow \quad \mathbf{B} = \mu_0 \mathbf{H}, \quad r(\mathbf{H} \cdot \nabla) \mathbf{n} = \mathbf{H} - \mathbf{n}(\mathbf{n} \cdot \mathbf{H}) \\ &= e \int \mathbf{n} (\nabla \cdot \mathbf{B}) d^3x - e \oint_S \mathbf{n} (\mathbf{B} \cdot \mathbf{n}_s) da \quad \Leftarrow \quad \text{integration by parts} \end{aligned}$$

$$\oint_S \mathbf{n} (\mathbf{B} \cdot \mathbf{n}_s) d a = \frac{g}{4 \pi} \oint \mathbf{n} d \Omega = 0 \quad \leftarrow \quad \mathbf{n} : \text{radially directed and no angular average} \quad \Rightarrow \quad \mathbf{L}_{em} = \frac{e g \mathbf{R}}{4 \pi R}$$

$$\nabla \cdot \mathbf{B} = g \delta(\mathbf{x} - \mathbf{R})$$

• In the collision process, the total angular momentum of the system (the particle & the EM field) is conserved, ie, the change in the angular momentum of the particle is balanced by the change in the EM angular momentum caused by the reversal of the direction \mathbf{R} .

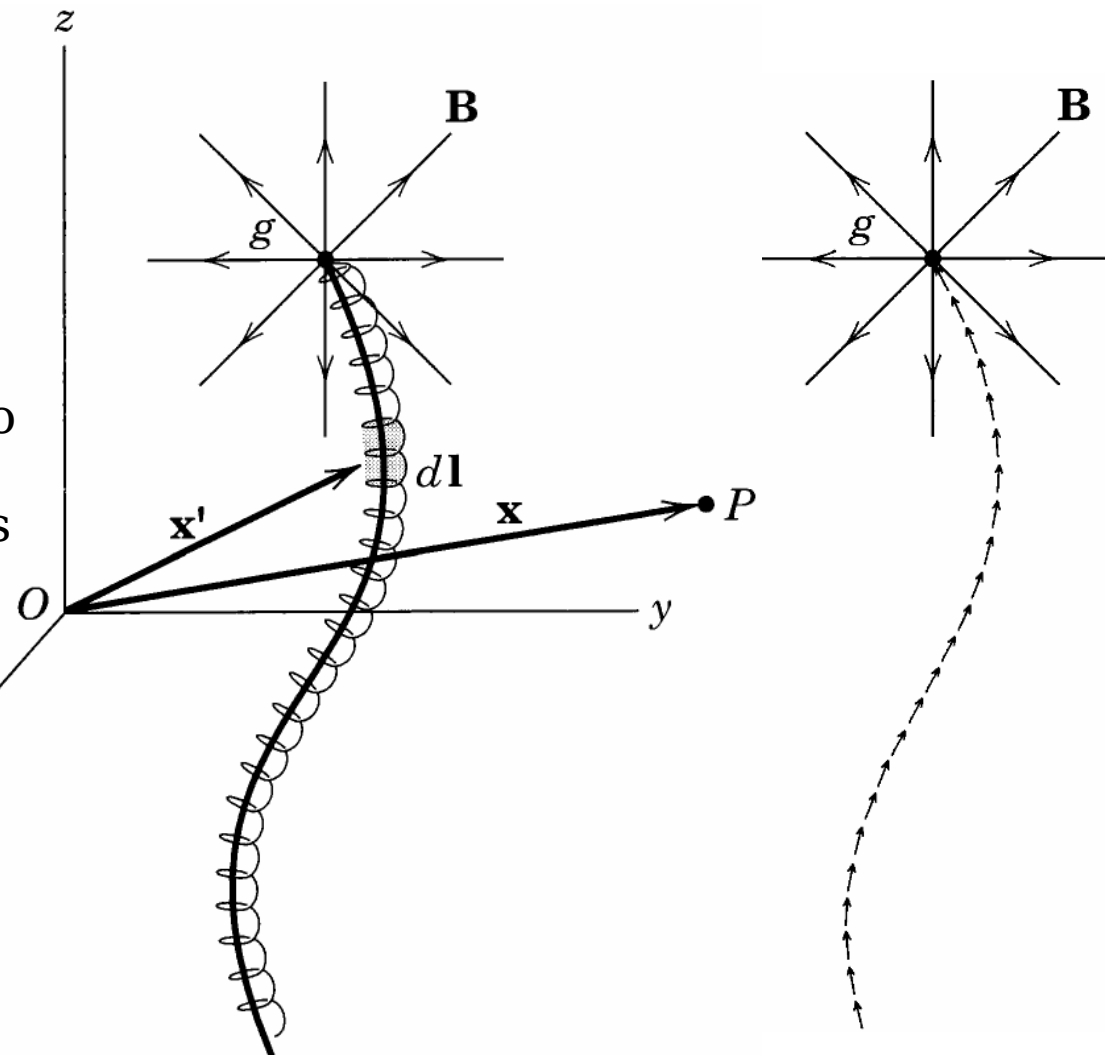
• Dirac's original argument: the standard Hamiltonian interaction

$$H_{int} = e \Phi - \frac{e}{m} \mathbf{p} \cdot \mathbf{A} + \frac{e^2}{2 m} \mathbf{A} \cdot \mathbf{A}$$

• The magnetic charge is imagined to be the end of a line of dipoles or a tightly wound solenoid that stretches off to infinity.

$$\bullet d \mathbf{A}(\mathbf{x}) = -\frac{\mu_0}{4 \pi} d \mathbf{m} \times \nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|}$$

$$\Rightarrow \mathbf{A}_L = -\frac{g}{4 \pi} \int_L d \boldsymbol{\ell} \times \nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|}$$



- the vector potential has a curl that is directed radially outward from the end of the string, varies inversely with distance squared, with total outward flux g .

- On the string the vector potential is singular, equivalent to a field \mathbf{B}' in the solenoid and bringing a return contribution of flux $-g$ in along the string.

- $\mathbf{B}_{\text{monopole}} = \nabla \times \mathbf{A} - \mathbf{B}' \leftarrow \mathbf{B}' : \text{exists only on the string (inside the solenoid)}$

- to describe the interaction of the electron with a magnetic *monopole*, rather than with a solenoid, Dirac asked that the electron never "see" the singular field \mathbf{B}' , so the electronic wave function to vanish along the string. g the singular [Not Good]

- a choice of different string positions is equivalent to different choices of gauge for the vector potential.

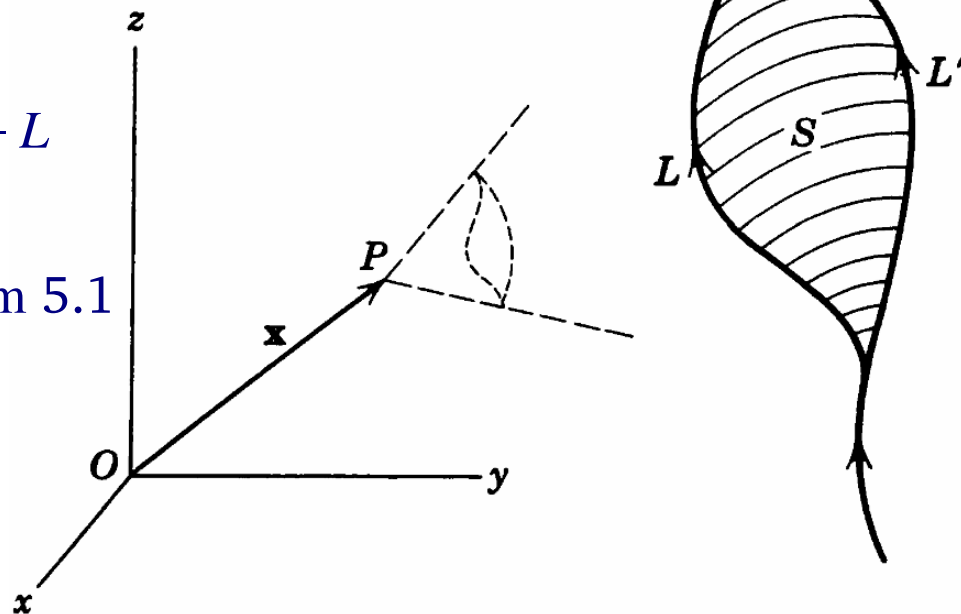
- The difference of the 2 vector potentials

$$\Delta \mathbf{A} = -\frac{g}{4\pi} \int_C d\boldsymbol{\ell} \times \nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|} \leftarrow C = L' - L$$

$$\Rightarrow \mathbf{A}_{L'}(\mathbf{x}) = \mathbf{A}_L(\mathbf{x}) + \frac{g}{4\pi} \nabla \Omega_C(\mathbf{x}) \leftarrow \text{Problem 5.1}$$

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla \chi$$

$$\Phi \rightarrow \Phi' = \Phi - \frac{1}{c} \frac{\partial \chi}{\partial t} \Rightarrow \chi = g \frac{\Omega_C}{4\pi}$$



- a change in the gauge of the EM potentials leaves the form of the Schrodinger equation invariant, provided the wave function is transformed according to

$$\left[\psi \rightarrow \psi' = \psi e^{i e \chi / \hbar} \Rightarrow \psi \rightarrow \psi' = \psi e^{i \frac{e g \Omega_c}{4 \pi \hbar}} \right] + \left[\Delta \Omega_c = 4 \pi \begin{array}{l} \text{as an electron} \\ \text{crosses the surface } S \end{array} \right]$$

$$\Rightarrow \frac{e g}{\hbar} = 2 \pi n \quad (n = 0, \pm 1, \pm 2, \dots) \quad \Leftarrow \quad \text{the wave function is single-valued}$$

- The argument follows from the general requirements of gauge invariance and single-valuedness of the wave function, independent of the location of the monopole's string.

6.13 Polarization Potentials (Hertz Vectors)

- use potentials as auxiliary fields to determine the fundamental EM fields.

- $\mathbf{D} = \epsilon \mathbf{E} + \mathbf{P}_{\text{ext}} \Leftrightarrow \mathbf{P}_{\text{ext}}, \mathbf{M}_{\text{ext}}$: external polarization densities

$$\mathbf{B} = \mu \mathbf{H} + \mu_0 \mathbf{M}_{\text{ext}}$$

$$\Rightarrow \mu \epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = +\mu \frac{\partial \mathbf{P}_{\text{ext}}}{\partial t} + \mu_0 \nabla \times \mathbf{M}_{\text{ext}} \Leftrightarrow \nabla \cdot \mathbf{A} + \mu \epsilon \frac{\partial \Phi}{\partial t} = 0$$

$$\mu \epsilon \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = -\frac{1}{\epsilon} \nabla \cdot \mathbf{P}_{\text{ext}}$$

$$\Rightarrow \mathbf{A} =: +\mu \frac{\partial \Pi_e}{\partial t} + \mu_0 \nabla \times \Pi_m \Leftrightarrow \Pi_e, \Pi_m : \text{vector polarization potentials}$$

$$\Phi =: -\frac{1}{\epsilon} \nabla \cdot \Pi_e$$

Hertz vectors

$$\Rightarrow \mu \frac{\partial}{\partial t} \left[\nabla^2 \Pi_e - \mu \epsilon \frac{\partial^2 \Pi_e}{\partial t^2} + \mathbf{P}_{\text{ext}} \right] + \mu_0 \nabla \times \left[\nabla^2 \Pi_m - \mu \epsilon \frac{\partial^2 \Pi_m}{\partial t^2} + \mathbf{M}_{\text{ext}} \right] = 0$$

$$\Rightarrow \mu \epsilon \frac{\partial^2 \Pi_e}{\partial t^2} - \nabla^2 \Pi_e = \mathbf{P}_{\text{ext}} - \frac{\mu_0}{\mu} \nabla \times \mathbf{V}$$

$$\mu \epsilon \frac{\partial^2 \Pi_m}{\partial t^2} - \nabla^2 \Pi_m = \mathbf{M}_{\text{ext}} + \frac{\partial \mathbf{V}}{\partial t} + \nabla \frac{\partial \xi}{\partial t}$$

$$\Rightarrow \mu \epsilon \frac{\partial^2 \Pi_e}{\partial t^2} - \nabla^2 \Pi_e = \mathbf{P}_{\text{ext}}$$

\Leftarrow the arbitrary functions \mathbf{V} & ξ may be removed by a gauge transformation [Problem 6.23]

$$\mu \epsilon \frac{\partial^2 \Pi_m}{\partial t^2} - \nabla^2 \Pi_m = \mathbf{M}_{\text{ext}}$$

$$\Rightarrow \mathbf{E} = \frac{1}{\epsilon} \nabla (\nabla \cdot \Pi_e) - \mu \frac{\partial^2 \Pi_e}{\partial t^2} - \mu_0 \nabla \times \frac{\partial \Pi_m}{\partial t} = -\mu_0 \nabla \times \frac{\partial \Pi_m}{\partial t} + \frac{1}{\epsilon} \nabla \times \nabla \times \Pi_e - \frac{1}{\epsilon} \mathbf{P}_{\text{ext}}$$

$$\mathbf{B} = \mu \nabla \times \frac{\partial \Pi_e}{\partial t} + \mu_0 \nabla \times \nabla \times \Pi_m$$

- For a simple case

$$\mathbf{M}_{\text{ext}} = \mathbf{m}(t) \delta(\mathbf{x} - \mathbf{x}_0) \Rightarrow \Pi_m(\mathbf{x}, t) = \frac{\mathbf{m}(t - \sqrt{\mu \epsilon} R)}{4 \pi R} \Leftarrow R = |\mathbf{x} - \mathbf{x}_0|$$

- Selected problems: 6.1, 6.3, 6.10, 6.14, 6.18, 6.21