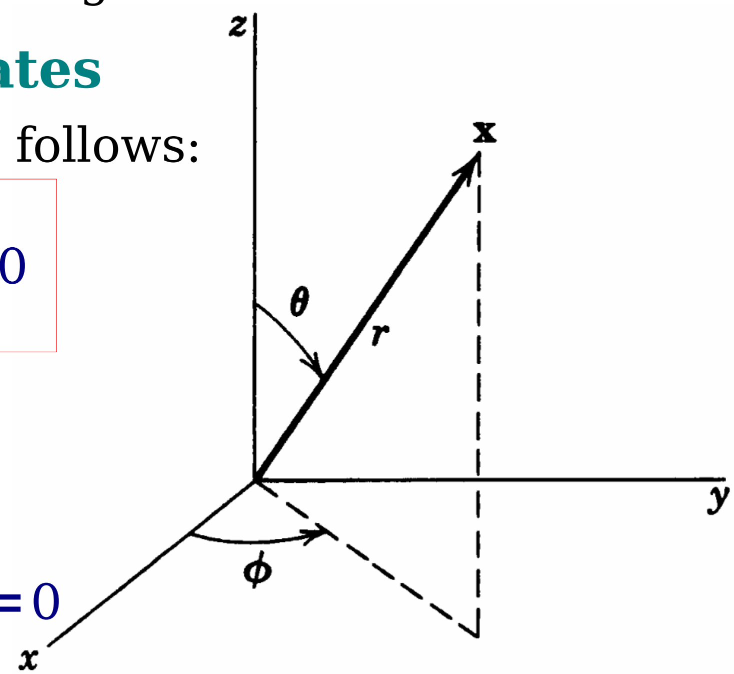
Chapter 3 Boundary-Value Problems in Electrostatics II

- Solutions of the Laplace equation are represented by expansions in series of the appropriate orthonormal functions in various geometries.

- The construction of Green functions in terms of orthonormal functions arises in the attempt to solve the Poisson equation in the various geometries.

§3.1 Laplace Equation in Spherical Coordinates



- In spherical coordinates, the Laplace equation is as follows:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=0$$

$$\Phi=\frac{U(r)}{r}P(\theta)Q(\phi)\quad\Leftarrow\quad\text{assuming a product form}$$

$$\Rightarrow\quad PQ\frac{d^2U}{dr^2}+\frac{UQ}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\frac{UP}{r^2\sin^2\theta}\frac{d^2Q}{d\phi^2}=0$$

$$\Rightarrow\quad \sin^2\theta\left[\frac{r^2}{U}\frac{d^2U}{dr^2}+\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\right]+\frac{1}{Q}\frac{d^2Q}{d\phi^2}=0$$

- The ϕ dependence of the equation has now been isolated in the last term:

$$\frac{1}{Q}\frac{d^2Q}{d\phi^2}=-m^2\quad\Leftarrow\text{constant}\quad\Rightarrow\quad Q=e^{\pm im\phi}\quad\Leftarrow\quad m:\text{ an integer for } Q \text{ being single-valued}$$

- By similar considerations

$$\frac{d^2 U}{d r^2} - \frac{\ell(\ell+1)}{r^2} U = 0 \quad \Leftarrow \quad \ell : \text{real constant} \quad \Rightarrow \quad \boxed{U = A r^{\ell+1} + B r^{-\ell}}$$

$$\frac{1}{\sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right] P = 0$$

§3.2 Legendre Equation and Legendre Polynomials

- $x = \cos\theta \quad \Rightarrow \quad \frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P = 0$

This equation is called the generalized Legendre equation, and its solutions are the associated Legendre functions.

- consider the ordinary Legendre differential equation with $m^2 = 0$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \ell(\ell+1)P = 0 \quad \text{where } -1 \le x \le 1 \quad \Rightarrow \quad P(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j$$

$$\Rightarrow \quad \sum_{j=0}^{\infty} \left\{(\alpha+j)(\alpha+j-1)a_j x^{\alpha+j-2} - [(\alpha+j)(\alpha+j+1) - \ell(\ell+1)]a_j x^{\alpha+j}\right\} = 0$$

$$\Rightarrow \quad a_{j+2} = a_j \frac{(\alpha+j)(\alpha+j+1) - \ell(\ell+1)}{(\alpha+j+1)(\alpha+j+2)} \quad \text{for } j \ge 0 \quad \Leftarrow \quad \begin{matrix} \alpha(\alpha-1) = 0 & \text{for } a_0 \ne 0 \\ \alpha(\alpha+1) = 0 & \text{for } a_1 \ne 0 \end{matrix}$$

$$\text{For } a_0 \ne 0 \quad \Rightarrow \quad \alpha = \begin{cases} 0 & \text{even power series of } x \\ 1 & \text{odd power series of } x \end{cases}$$

- The two relations of α are equivalent, so choose one of a_0 and a_1 being nonzero.

- the series converges for $x^2 < 1$, regardless of the value of ℓ.

the series diverges at $x=\pm1$, unless it terminates.

- Since we want a solution that is finite, we demand that the series terminate.
- Since α and j are positive integers or zero, the recurrence relation will terminate only if ℓ *is zero or a positive integer*.
- If ℓ is even (odd), then only the α=0 (α=1) series terminates.
- The polynomials in each case have x^ℓ as their highest power of x, the next highest being $x^{\ell-2}$, down to $x^0(x)$ for ℓ even (odd).
- these polynomials are normalized to be unity at $x=\pm 1$ and are called the *Legendre polynomials* of order ℓ,

$$P_0(x)=1,\quad P_1(x)=x,\quad P_2(x)=\frac{1}{2}(3x^2-1),\quad \cdots$$

$$P_3(x)=\frac{1}{2}(5x^3-3x),\quad p_4(x)=\frac{1}{8}(35x^4-30x^2+3)$$

$$\Rightarrow \quad P_\ell(x)=\frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$

Rodrigues' formula

- The Legendre polynomials form a complete orthogonal set of functions on the interval $-1\le x\le 1$.

$$\int_{-1}^{1} P_{\ell'}\left\{\frac{d}{dx}\left[(1-x^2)\frac{dP_\ell}{dx}\right]+\ell(\ell+1)P_\ell\right\}dx=0$$

$$\Rightarrow \quad \int_{-1}^{1}\left[(x^2-1)\frac{dP_\ell}{dx}\frac{dP_{\ell'}}{dx}+\ell(\ell+1)P_\ell P_{\ell'}\right]dx=0$$

$$\ell \leftrightarrow \ell' \Rightarrow [\ell(\ell+1) - \ell'(\ell'+1)] \int_{-1}^{1} P_{\ell'} P_\ell \, dx = 0 \Rightarrow \int_{-1}^{1} P_{\ell'} P_\ell \, dx \begin{cases} = 0 & \text{for } \ell \neq \ell' \\ \neq 0 & \text{for } \ell = \ell' \end{cases}$$

- Use Rodrigues' formula to determine the value for $\ell = \ell'$

$$N_\ell \equiv \int_{-1}^{1} P_\ell^2 \, dx = \frac{1}{4^\ell (\ell!)^2} \int_{-1}^{1} \frac{d^\ell}{dx^\ell} (x^2-1)^\ell \frac{d^\ell}{dx^\ell} (x^2-1)^\ell \, dx$$

$$= \frac{(-1)^\ell}{4^\ell (\ell!)^2} \int_{-1}^{1} (x^2-1)^\ell \frac{d^{2\ell}}{dx^{2\ell}} (x^2-1)^\ell \, dx = \frac{(2\ell)!}{4^\ell (\ell!)^2} \int_{-1}^{1} (1-x^2)^\ell \, dx$$

$$(1-x^2)^\ell = (1-x^2)(1-x^2)^{\ell-1} = (1-x^2)^{\ell-1} + \frac{x}{2\ell} \frac{d}{dx} (1-x^2)^\ell$$

$$\Rightarrow \quad N_\ell = \frac{2\ell-1}{2\ell} N_{\ell-1} + \frac{(2\ell-1)!}{4^\ell (\ell!)^2} \int_{-1}^{1} x \, d(1-x^2)^\ell = \frac{2\ell-1}{2\ell} N_{\ell-1} - \frac{1}{2\ell} N_\ell$$

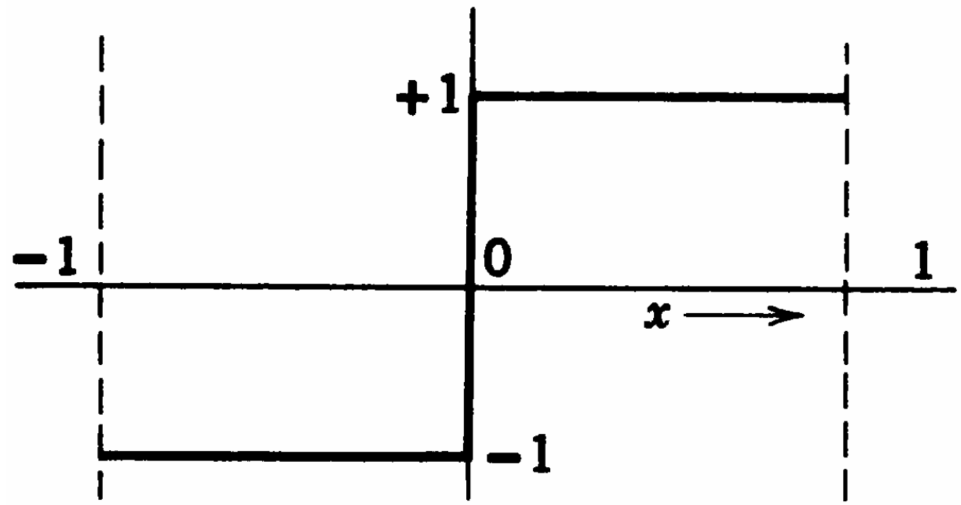
$$(2\ell+1) N_\ell = (2\ell-1) N_{\ell-1} \Rightarrow \text{independent of } \ell \quad \Rightarrow \quad N_\ell = \frac{2}{2\ell+1} \quad \Leftarrow N_0 = 2 \quad \Leftarrow P_0 = 1$$

$$\Rightarrow \quad \boxed{\int_{-1}^{1} P_\ell P_{\ell'} \, dx = \frac{2}{2\ell+1} \delta_{\ell\ell'} \quad \begin{pmatrix} \text{orthogonality} \\ \text{condition} \end{pmatrix}} \quad \Rightarrow \quad U_\ell = \sqrt{\frac{2\ell+1}{2}} P_\ell$$

- For any function $f(x)$ on the interval $-1 \leq x \leq 1$

$$f(x) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(x) \quad \Leftarrow \quad A_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} f(x) P_\ell(x) \, dx$$

Ex: consider $f(x) = \begin{cases} +1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \end{cases}$



$$A_\ell = \frac{2\ell+1}{2}\left(\int_0^1 P_\ell \, \mathrm{d}x - \int_{-1}^0 P_\ell \, \mathrm{d}x\right)$$

$$= (2\ell+1)\int_0^1 P_\ell \, \mathrm{d}x \quad \left[\begin{array}{l}\text{only odd } \ell \\ \text{coefficients survive}\end{array}\right]$$

$$= (-2)^{\frac{1-\ell}{2}} \frac{(2\ell+1)(\ell-2)!!}{2[(\ell+1)/2]!} \Leftarrow$$ from Rodrigues' formula, and $(2n+1)!! = (2n+1)(2n-1)(2n-3)\cdots\times 5\times 3\times 1$

$$\Rightarrow \quad f(x) = \frac{3}{2}P_1 - \frac{7}{8}P_3 + \frac{11}{16}P_5 - \cdots$$

- Some useful recurrence relations

$$\frac{\mathrm{d}P_{\ell+1}}{\mathrm{d}x} - \frac{\mathrm{d}P_{\ell-1}}{\mathrm{d}x} - (2\ell+1)P_\ell = 0, \qquad (\ell+1)P_{\ell+1} - (2\ell+1)xP_\ell + \ell P_{\ell-1} = 0 \qquad (*)$$

$$\frac{\mathrm{d}P_{\ell+1}}{\mathrm{d}x} - x\frac{\mathrm{d}P_\ell}{\mathrm{d}x} - (\ell+1)P_\ell = 0, \qquad (x^2-1)\frac{\mathrm{d}P_\ell}{\mathrm{d}x} - \ell x P_\ell + \ell P_{\ell-1} = 0$$

- $P_\ell(x) = \sum_{n=0}^{[\ell/2]} (-1)^n \frac{(2\ell-2n)!}{2^\ell n!(\ell-n)!(\ell-2n)!} x^{\ell-2n}$ $\Leftarrow$ *Mathematical methods for Physicists*, Arfken

Ex: consider $I_1 = \int_{-1}^{1} x\, P_\ell\, P_{\ell'}\, \mathrm{d}x = \dfrac{1}{2\ell+1} \int_{-1}^{1} P_{\ell'} \left[(\ell+1)\, P_{\ell+1} + \ell\, P_{\ell-1} \right] \mathrm{d}x \quad \Leftarrow \quad (*)$

$$= \begin{cases} \dfrac{2(\ell+1)}{(2\ell+1)(2\ell+3)}, & \ell' = \ell+1 \\[2ex] \dfrac{2\ell}{(2\ell-1)(2\ell+1)}, & \ell' = \ell-1 \end{cases} \quad \Leftarrow \quad (\text{orthogonality condition})$$

Similarly,

$$\int_{-1}^{1} x^2\, P_\ell\, P_{\ell'}\, \mathrm{d}x = \begin{cases} \dfrac{2(\ell+1)(\ell+2)}{(2\ell+1)(2\ell+3)(2\ell+5)}, & \ell' = \ell+2 \\[2ex] \dfrac{2(2\ell^2+2\ell-1)}{(2\ell-1)(2\ell+1)(2\ell+3)}, & \ell' = \ell \end{cases} \quad \Leftarrow \quad \text{assuming} \quad \ell' \geq \ell$$

§3.3 Boundary- Value Problems with Azimuthal Symmetry

- The general solution for a problem possessing azimuthal symmetry $m = 0$

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} [A_\ell r^\ell + B_\ell r^{-(\ell+1)}] P_\ell(\cos\theta) \Leftarrow$$

determining A_ℓ , B_ℓ , from the boundary condition

On the symmetric axis z $\Rightarrow$
$$\begin{cases} \Phi(\theta=0) = \sum [A_\ell r^\ell + B_\ell r^{-(\ell+1)}] \\ \Phi(\theta=\pi) = \sum (-1)^\ell [A_\ell r^\ell + B_\ell r^{-(\ell+1)}] \end{cases}$$

- This expansion provides a means of obtaining the solution of potential problems from a knowledge of the potential in a limited domain, i.e., on the symmetry axis.

Ex: let $V(\theta)$ be the potential on the surface of a sphere of radius a, find the potential inside the sphere.

No charge at the origin $\Rightarrow$ $B_\ell = 0 \Rightarrow V(\theta) = \sum_{\ell=0}^{\infty} A_\ell a^\ell P_\ell(\cos\theta)$

$$\Rightarrow A_\ell = \frac{2\ell+1}{2a^\ell}\int_0^\pi V(\theta) P_\ell(\cos\theta)\sin\theta\, d\theta \quad \Rightarrow \quad \text{let } V(\theta) = \begin{cases} +V, & (0 \le \theta < \pi/2) \\ -V, & (\pi/2 < \theta \le \pi) \end{cases}$$

$$\Rightarrow \Phi(r,\theta) = V\left[\frac{3}{2}\frac{r}{a}P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \frac{11}{16}\left(\frac{r}{a}\right)^5 P_5(\cos\theta) + \cdots\right] \quad \text{for } r < a$$

$$\Rightarrow \Phi(r,\theta) = V\left[\frac{3}{2}\left(\frac{a}{r}\right)^2 P_1(\cos\theta) - \frac{7}{8}\left(\frac{a}{r}\right)^4 P_3(\cos\theta) + \frac{11}{16}\left(\frac{a}{r}\right)^6 P_5(\cos\theta) + \cdots\right] \quad \text{for } r > a$$

- For the problem of the hemispheres at equal and opposite potentials. We have

$$\Phi(z=r)=V\left[1-\frac{r^2-a^2}{r\sqrt{r^2+a^2}}\right]=\frac{V}{\sqrt{\pi}}\sum_{j=1}^{\infty}(-1)^{j-1}\frac{(2j-1/2)\,\Gamma(2j-1/2)}{j!}\left(\frac{a}{r}\right)^{2j}$$

$$\Rightarrow\quad \Phi(r,\theta)=\frac{V}{\sqrt{\pi}}\sum_{j=1}^{\infty}(-1)^{j-1}\frac{(2j-1/2)\,\Gamma(2j-1/2)}{j!}\left(\frac{a}{r}\right)^{2j}P_{2j-1}(\cos\theta)\qquad =(2.27)\;=(3.36)$$

- the potential at **x** due to a unit point charge at **x'**:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\sum_{\ell=0}^{\infty}\frac{r_<^{\ell}}{r_>^{\ell+1}}P_\ell(\cos\gamma)\quad\Leftarrow\quad r_<=\min(|\mathbf{x}|,|\mathbf{x}'|),\quad r_>=\max(|\mathbf{x}|,|\mathbf{x}'|)$$

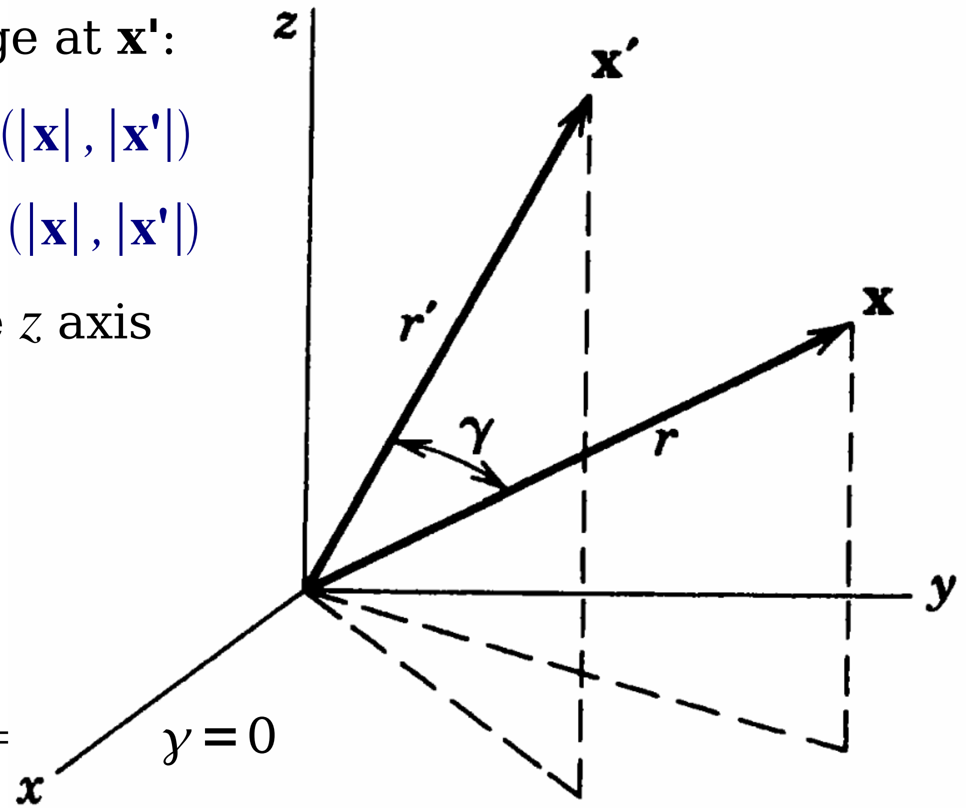
Proof: rotating axes so that **x'** lies along the z axis

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\sum_{\ell=0}^{\infty}\left[A_\ell r^{\ell}+B_\ell r^{-(\ell+1)}\right]P_\ell(\cos\gamma)$$

$$\to\sum_{\ell=0}^{\infty}\left[A_\ell r^{\ell}+B_\ell r^{-(\ell+1)}\right]\quad\Leftarrow\quad\gamma=0$$

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}}\to\frac{1}{|r-r'|}\quad\Leftarrow\quad\gamma=0$$

$$\to\sum_{\ell=0}^{\infty}\frac{r_<^{\ell}}{r_>^{\ell+1}},$$ then multiply each term by $P_\ell(\cos\gamma)$ for points off the axis. QED



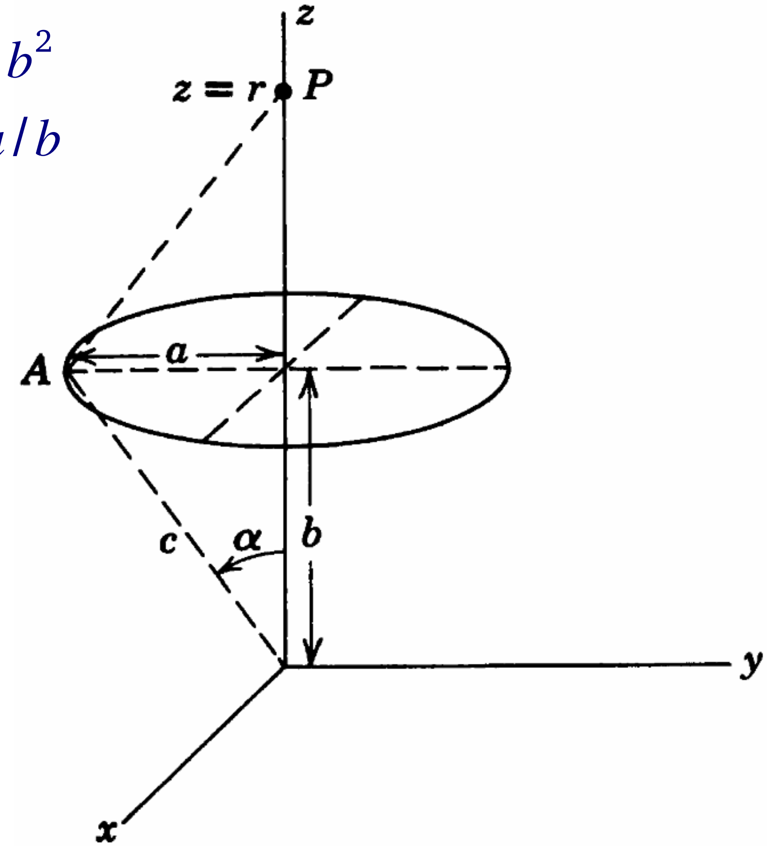
- the potential due to a total charge uniformly distributed around a circular ring

$$\Phi(z=r)=\frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{r^2+c^2-2cr\cos\alpha}} \Leftarrow \quad c^2=a^2+b^2, \quad \cos\alpha=a/b$$

$$=\begin{cases}\dfrac{q}{4\pi\epsilon_0}\displaystyle\sum_{\ell=0}^{\infty}\frac{c^\ell}{r^{\ell+1}}P_\ell(\cos\alpha) & r>c\\[2ex] \dfrac{q}{4\pi\epsilon_0}\displaystyle\sum_{\ell=0}^{\infty}\frac{r^\ell}{c^{\ell+1}}P_\ell(\cos\alpha) & r<c\end{cases}$$

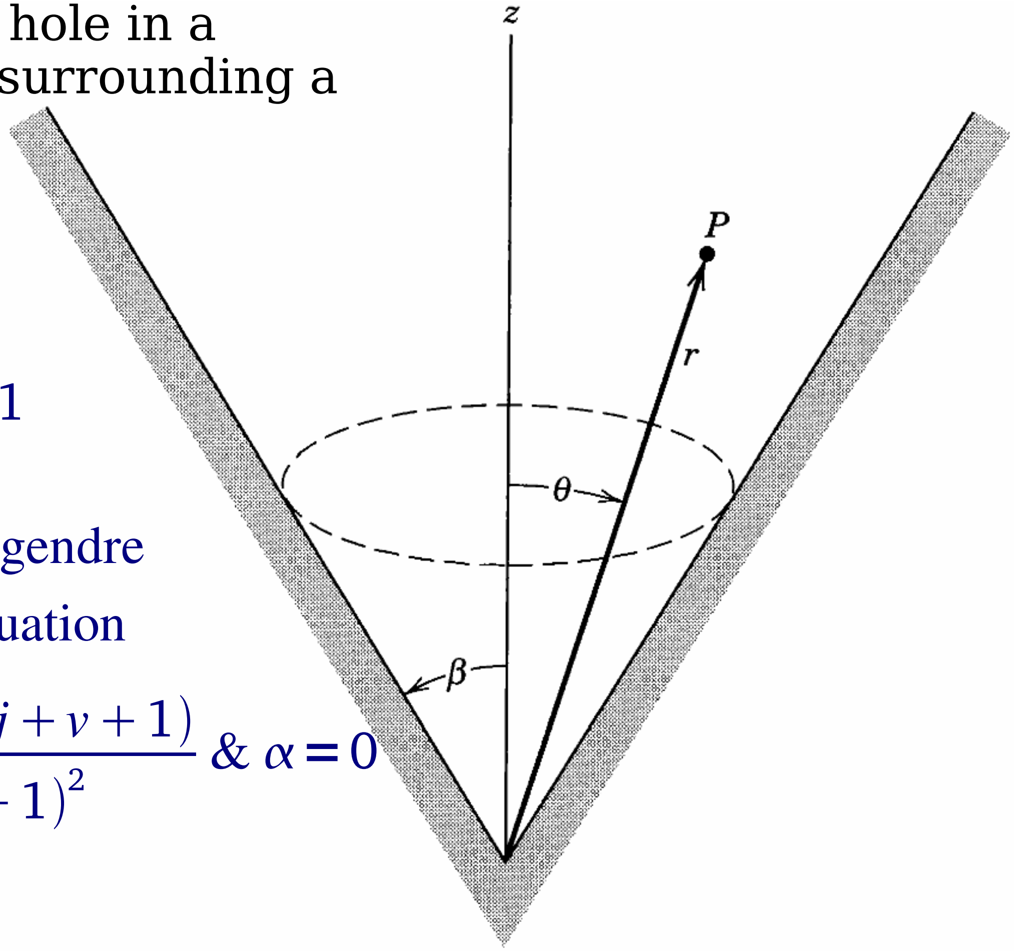
$$\Rightarrow \quad \Phi(r,\theta)=\frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\alpha)P_\ell(\cos\theta)$$

where $r_<=\min(r,c)$, $r_>=\max(r,c)$



§3.4 Behavior of Fields in a Conical Hole or Near a Sharp Point

- For $\beta<\pi/2$, the region is a deep conical hole in a conductor. For $\beta>\pi/2$, the region is that surrounding a pointed conical conductor.



- seek solutions finite and single-valued on the range of $x=\cos\theta$

$$\xi \equiv \frac{1}{2}(1-x) \quad \Leftarrow \quad x \equiv \cos\theta \quad \Rightarrow \quad \cos\beta < x < 1$$

$$\Rightarrow \frac{d}{d\xi}\left[\xi(1-\xi)\frac{dP}{d\xi}\right] + \nu(\nu+1)P = 0 \quad \Leftarrow \text{Legendre equation}$$

$$\Rightarrow \quad P(\xi) = \xi^{\alpha}\sum_{j=0}^{\infty} a_j \xi^j \quad \Leftarrow \quad \frac{a_{j+1}}{a_j} = \frac{(j-\nu)(j+\nu+1)}{(j+1)^2} \ \& \ \alpha = 0$$

$$\text{set } a_0 = 1 \quad \Rightarrow \quad P(\xi=0) = 1$$

$$\Rightarrow \quad P_\nu(\xi) = 1 + \frac{(-\nu)(\nu+1)}{1!\,1!}\xi + \frac{(-\nu)(-\nu+1)(\nu+1)(\nu+2)}{2!\,2!}\xi^2 + \cdots$$

- if ν is zero or a positive integer the series is exactly the Legendre *polynomials*.

- For ν not being an integer, the series is called a *Legendre function of the 1st kind and order* ν.

- hypergeometric function $_2F_1(a,b;c;z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots$

$$\Rightarrow \quad P_\nu(x) = {}_2F_1\left(-\nu, \nu+1; 1; \frac{1-x}{2}\right)$$

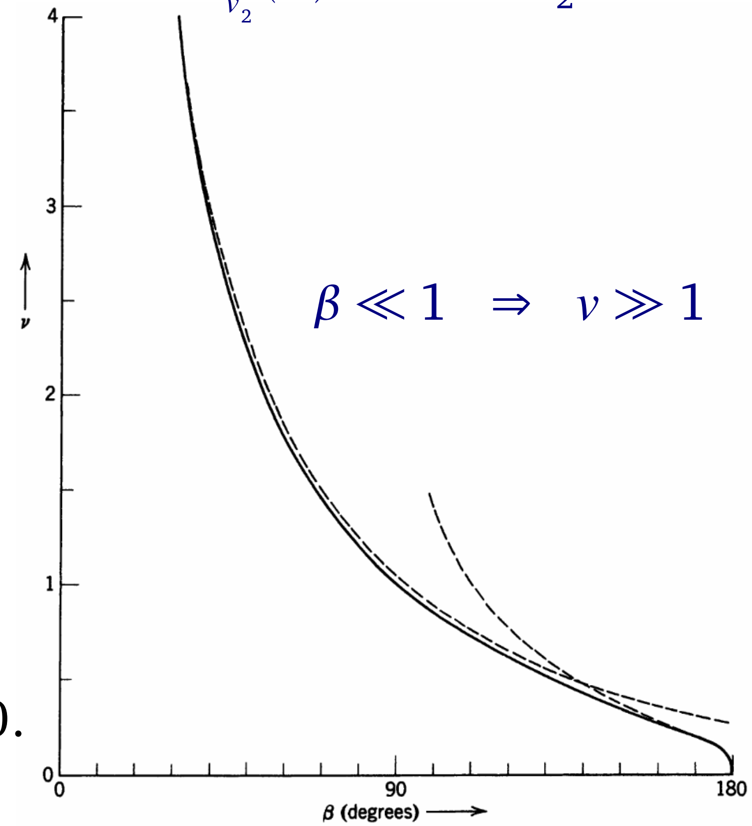
- The basic solution to the problem: $A\, r^\nu P_\nu(\cos\theta) \quad \Leftarrow \quad \nu > 0$ for finite at the origin

- the potential vanishes at $\theta=\beta$: $P_\nu(\cos\beta) = 0$ an eigenvalue condition for ν.

$$\Phi(r,\theta) = \sum_{k=1}^{\infty} A_k r^{\nu_k} P_{\nu_k}(\cos\theta) \quad \Leftarrow \quad \begin{cases} \cos\beta \text{ is the 1st zero for } P_{\nu_1}(x) & \text{for } \nu = \nu_1 \\ \cos\beta \text{ is the 2nd zero for } P_{\nu_2}(x) & \text{for } \nu = \nu_2 \\ \ldots \end{cases}$$

$$r \to 0 \quad \Rightarrow \quad \Phi \simeq A\, r^\nu P_\nu(\cos\theta) \quad \Leftarrow \quad \nu : \text{the smallest root}$$

$$\Rightarrow \quad \begin{cases} E_r = -\frac{\partial \Phi}{\partial r} \simeq \nu A\, r^{\nu-1} P_\nu(\cos\theta) \\ E_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} \simeq A\, r^{\nu-1} P'_\nu(\cos\theta)\sin\theta \\ \sigma(r) = -\epsilon_0 E_\theta\big|_{\theta=\beta} \simeq -\epsilon_0 A\, r^{\nu-1} P'_\nu(\cos\beta)\sin\beta \end{cases}$$



$$\beta \ll 1 \quad \Rightarrow \quad \nu \gg 1$$

- The fields and charge density all vary as $r^{\nu-1}$ as $r \to 0$.

- An approximate expression for ν can be from the Bessel function

$$P_\nu(\cos\theta) \simeq J_0\left[(2\nu+1)\sin\frac{\theta}{2}\right] \quad \Leftarrow \quad \text{for large } \nu \text{ and } \theta<1$$

- The 1st zero of $J_0(x)$ is at x=2.405 $\Rightarrow \quad \nu \simeq \dfrac{2.405}{\beta} - \dfrac{1}{2}$
- $\beta \to 0 \quad \Rightarrow \quad |\mathbf{E}|,\, \sigma \propto r^{\nu-1} \to 0$ small fields & little charge deep in a conical hole
- $\beta = \dfrac{\pi}{2} \quad \Rightarrow \quad \nu = 1 \quad \Rightarrow \quad \sigma \propto 1\,(\text{const})$ (plane)
- $\beta > \dfrac{\pi}{2} \quad \Rightarrow \quad \nu < 1 \quad \Rightarrow \quad$ singular at $r = 0$
- $\beta \to \pi \quad \Rightarrow \quad \nu \to 0 \quad \Rightarrow \quad \nu \simeq \left[2\ln\dfrac{2}{\pi-\beta}\right]^{-1} \quad \Rightarrow \quad \begin{cases} \nu \simeq 0.2 & \Leftarrow \quad \pi-\beta \simeq 10^\circ \\ \nu \simeq 0.1 & \Leftarrow \quad \pi-\beta \simeq 1^\circ \end{cases}$
- the fields near a narrow conical point vary $r^{-1+\epsilon}$, $\epsilon \ll 1$, and very high fields exist around the point.

§3.5 Associated Legendre Functions and the Spherical Harmonics

- The general potential problem can, however, have azimuthal variations.
- associated Legendre function: generalization of Legendre polynomial

$$P_\ell^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_\ell(x) \quad \Leftarrow \quad \ell \in \mathbb{N} \cup 0, \quad m = -\ell, \cdots, 0, \cdots, \ell$$

$$= \frac{(-1)^m}{2^\ell \ell!} (1-x^2)^{\frac{m}{2}} \frac{d^{m+\ell}}{dx^{m+\ell}} (x^2-1)^\ell \quad \Leftarrow \quad \text{Rodrigues' formula}$$

$$\Rightarrow \quad P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)$$

$$\Rightarrow \quad \int_{-1}^{1} P_{\ell'}^m(x) P_\ell^m(x)\, dx = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell'\ell}$$

- *spherical harmonics* (tesseral harmonics)

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta) e^{im\phi} \quad \Rightarrow \quad Y_{\ell,-m}(\theta,\phi) = (-1)^m Y_{\ell m}^*(\theta,\phi)$$

$$\Rightarrow \quad \int_0^{2\pi} \int_0^{\pi} Y_{\ell' m'}^*(\theta,\phi) Y_{\ell m}(\theta,\phi) \sin\theta\, d\theta\, d\phi = \delta_{\ell'\ell} \delta_{m'm} \quad \Leftarrow \quad \text{normalization orthogonality}$$

$$\Rightarrow \quad \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta',\phi') Y_{\ell m}(\theta,\phi) = \delta(\phi-\phi') \delta(\cos\theta - \cos\theta') \quad \Leftarrow \quad \text{completeness}$$

- $\ell=0 \quad Y_{00}=\dfrac{1}{\sqrt{4\pi}}$

$\ell=1 \quad \begin{cases} Y_{11}=-\sqrt{\dfrac{3}{8\pi}}\sin\theta\, e^{i\phi} \\ Y_{10}=\sqrt{\dfrac{3}{4\pi}}\cos\theta \end{cases}$

$\ell=2 \quad \begin{cases} Y_{22}=\dfrac{1}{4}\sqrt{\dfrac{15}{2\pi}}\sin^2\theta\, e^{2i\phi} \\ Y_{21}=-\sqrt{\dfrac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi} \\ Y_{20}=\dfrac{1}{2}\sqrt{\dfrac{5}{4\pi}}(3\cos^2\theta-1) \end{cases}$

$\ell=3 \quad \begin{cases} Y_{33}=-\dfrac{1}{4}\sqrt{\dfrac{35}{4\pi}}\sin^3\theta\, e^{3i\phi} & Y_{32}=\dfrac{1}{4}\sqrt{\dfrac{105}{2\pi}}\sin^2\theta\cos\theta\, e^{2i\phi} \\ Y_{31}=-\dfrac{1}{4}\sqrt{\dfrac{21}{4\pi}}\sin\theta(5\cos^2\theta-1)\, e^{i\phi} & Y_{30}=\dfrac{1}{2}\sqrt{\dfrac{7}{4\pi}}(5\cos^3\theta-3\cos\theta) \end{cases}$

- For $m=0$, $\quad Y_{\ell 0}(\theta,\phi)=\sqrt{\dfrac{2\ell+1}{4\pi}}P_\ell(\theta)$

- For an arbitrary function

$$g(\theta,\phi)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}A_{\ell m}Y_{\ell m}(\theta,\phi) \quad \Leftarrow \quad A_{\ell m}=\int g(\theta,\phi)\,Y^*_{\ell m}(\theta,\phi)\,\mathrm{d}\Omega$$

- the expansion for $\theta=0$

$$[g(\theta,\phi)]_{\theta=0} = \sum_{\ell=0}^{\infty} \sqrt{\frac{2\ell+1}{4\pi}} A_{\ell 0} \quad \Leftarrow \quad A_{\ell 0} = \sqrt{\frac{2\ell+1}{4\pi}} \int g(\theta,\phi) P_\ell(\cos\theta) \, \mathrm{d}\Omega \qquad (*)$$

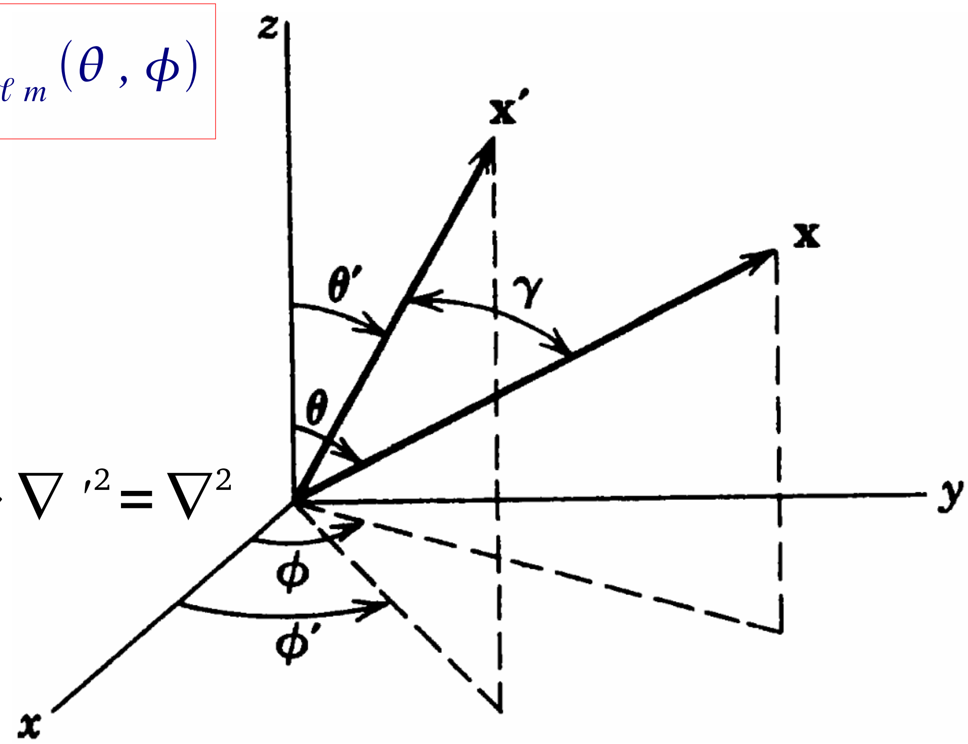
All terms in the series with $m \neq 0$ vanish at $\theta = 0$

- The general solution $$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} [A_{\ell m} r^{\ell} + B_{\ell m} r^{-(\ell+1)}] Y_{\ell m}(\theta,\phi)$$

§3.6 Addition Theorem for Spherical Harmonics

- For $\mathbf{x}=(r,\theta,\phi)$, $\mathbf{x}'=(r',\theta',\phi')$ $\Rightarrow$ $\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')$

$$\Rightarrow \quad P_\ell(\cos\gamma)=\frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\phi')\,Y_{\ell m}(\theta,\phi)$$



Proof: if let **x'** be on the z-axis

$$\Rightarrow \quad \nabla'^2 P_\ell(\cos\gamma)+\frac{\ell(\ell+1)}{r^2}P_\ell(\cos\gamma)=0$$

If rotate the axis to a new place

∇^2 :scalar operator $\Rightarrow$ invariant under rotation $\Rightarrow$ $\nabla'^2=\nabla^2$

$$\Rightarrow \quad \nabla^2 P_\ell(\cos\gamma)+\frac{\ell(\ell+1)}{r^2}P_\ell(\cos\gamma)=0$$

$\Rightarrow$ P_ℓ is a speherical harmonic of ℓ

$$\Rightarrow \quad P_\ell=\sum_{m=-\ell}^{\ell} A_m(\theta',\phi')\,Y_{\ell m}(\theta,\phi) \quad \Leftarrow \quad A_m(\theta',\phi')=\int Y^*_{\ell m}(\theta,\phi)\,P_\ell(\cos\gamma)\,d\Omega$$

$g(\gamma,\beta)=\sqrt{\dfrac{4\pi}{2\ell+1}}\,Y^*_{\ell m}(\theta,\phi)$ expanded with $Y_{\ell m'}(\gamma,\beta)$ (the prime axis) and $m'=0$

using $(*)$ $\Rightarrow$ $A_m(\theta',\phi')=\dfrac{4\pi}{2\ell+1}Y^*_{\ell m}(\theta,\phi)\Big|_{\gamma=0}=\dfrac{4\pi}{2\ell+1}Y^*_{\ell m}(\theta',\phi')$ QED

- Another theorem

$$P_\ell(\cos\gamma) = P_\ell(\cos\theta)\,P_\ell(\cos\theta') + 2\sum_{m=1}^{\ell}\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(\cos\theta)\,P_\ell^m(\cos\theta')\cos m(\phi-\phi')$$

- $\gamma = 0 \quad \Rightarrow \quad \sum_{m=-\ell}^{\ell}\left|Y_{\ell m}(\theta,\phi)\right|^2 = \frac{2\ell+1}{4\pi}$

- A useful application of the addition theorem

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\gamma) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y_{\ell m}^*(\theta',\phi')\,Y_{\ell m}(\theta,\phi)$$

§3.7 Laplace Equation in Cylindrical Coordinates; Bessel Functions

- The Laplace equations: $\dfrac{\partial^2 \Phi}{\partial \rho^2}+\dfrac{1}{\rho}\dfrac{\partial \Phi}{\partial \rho}+\dfrac{1}{\rho^2}\dfrac{\partial^2 \Phi}{\partial \phi^2}+\dfrac{\partial^2 \Phi}{\partial z^2}=0 \quad \Rightarrow \quad \Phi = R(\rho)\,Q(\phi)\,Z(z)$

$$\Rightarrow \quad \frac{d^2 Z}{d z^2}-k^2 Z=0, \qquad \frac{d^2 Q}{d\phi^2}+\nu^2 Q=0, \qquad \frac{d^2 R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}+\left(k^2-\frac{\nu^2}{\rho^2}\right)R=0$$

$$\Rightarrow \quad \boxed{Z(z)=e^{\pm k z}}, \quad \boxed{Q(\phi)=e^{\pm i \nu \phi}} \quad \Leftarrow \quad \nu \in \mathbb{Z}, \quad k \in \mathbb{R}(+)$$

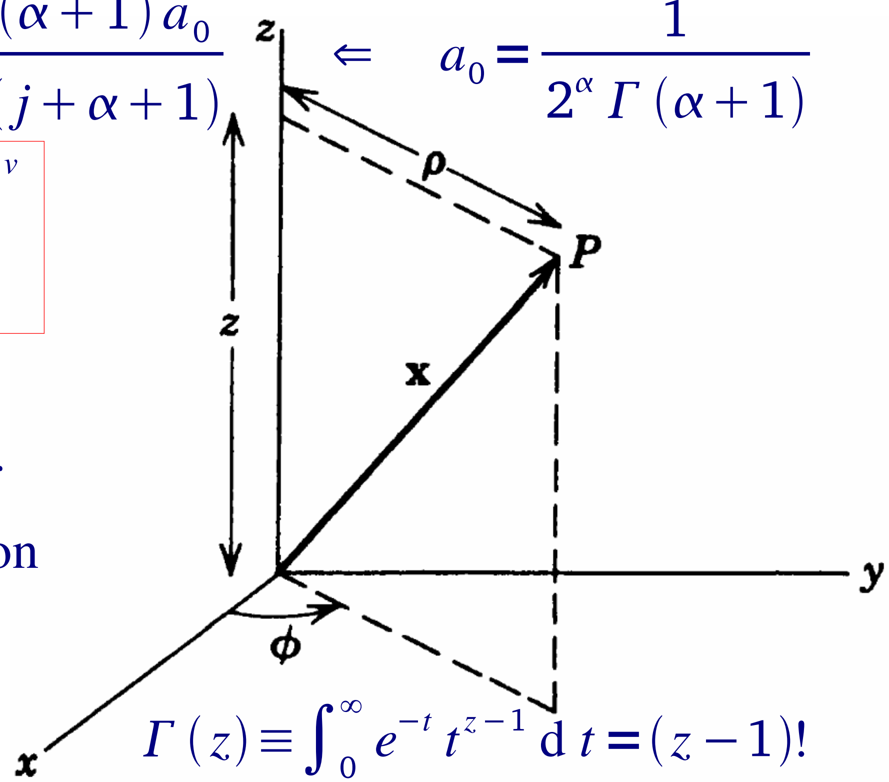
$$x \equiv k\rho \quad \Rightarrow \quad \boxed{\frac{d^2 R}{d x^2}+\frac{1}{x}\frac{dR}{dx}+\left(1-\frac{\nu^2}{x^2}\right)R=0} \quad \Leftarrow \quad \text{Bessel equation} \quad \Rightarrow \quad R(x)=x^\alpha \sum_{j=0}^{\infty} a_j x^j$$

$$\Rightarrow \quad \alpha=\pm\nu, \quad a_{\text{odd}}=0, \quad a_{2j}=-\frac{a_{2j-2}}{4j(j+\alpha)}=\frac{(-1)^j\,\Gamma(\alpha+1)\,a_0}{2^{2j}\,j!\,\Gamma(j+\alpha+1)} \quad \Leftarrow \quad a_0=\frac{1}{2^\alpha\,\Gamma(\alpha+1)}$$

$$\Rightarrow \quad \text{2 solutions:} \quad \boxed{J_{\pm\nu}(x)=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\,\Gamma(j\pm\nu+1)}\left(\frac{x}{2}\right)^{2j\pm\nu}}$$

$$\begin{cases} \nu \notin \mathbb{Z} \quad \Rightarrow \quad J_{\pm\nu} \ \text{are linearly independent} \\ \nu \in \mathbb{Z} \quad \Rightarrow \quad J_{-\nu}=(-1)^\nu J_\nu \quad \Leftarrow \quad \text{need another linear independent solution} \end{cases}$$

$$\Rightarrow \quad \boxed{N_\nu=\frac{J_\nu \cos\nu\pi - J_{-\nu}}{\sin\nu\pi}} \quad \Leftarrow \quad \text{Neumann function}$$



$$\Gamma(z)\equiv\int_0^\infty e^{-t}\,t^{z-1}\,dt=(z-1)!$$

- replace $J_{\pm\nu}$ with J_ν and N_ν no matter if ν is an integer or not.
- Hankel functions: $H_\nu^{(1)} \equiv J_\nu + i\, N_\nu\,, \quad H_\nu^{(2)} \equiv J_\nu - i\, N_\nu$
- $\Omega_\nu \in \{J_\nu\,, N_\nu\,, H_\nu^{(1)}\,, H_\nu^{(2)}\} \quad \Rightarrow \quad \Omega_{\nu-1} + \Omega_{\nu+1} = \dfrac{2\,\nu}{x}\,\Omega_\nu\,, \quad \Omega_{\nu-1} - \Omega_{\nu+1} = 2\,\dfrac{d\,\Omega_\nu}{d\,x} \quad (\#)$
- $x \ll 1 \quad \Rightarrow \quad J_\nu \to \dfrac{1}{\Gamma(\nu+1)}\left(\dfrac{x}{2}\right)^\nu, \quad N_\nu \to \begin{cases} \dfrac{2}{\pi}\left[\ln\dfrac{x}{2} + 0.5772\cdots\right], & \nu = 0 \\ -\dfrac{\Gamma(\nu)}{\pi}\left(\dfrac{2}{x}\right)^\nu, & \nu > 0 \end{cases}$

  $x \gg 1, \nu \quad \Rightarrow \quad J_\nu \to \sqrt{\dfrac{2}{\pi\, x}}\cos\left(x - \dfrac{\nu\,\pi}{2} - \dfrac{\pi}{4}\right), \quad N_\nu \to \sqrt{\dfrac{2}{\pi\, x}}\sin\left(x - \dfrac{\nu\,\pi}{2} - \dfrac{\pi}{4}\right)$
- $J_\nu(x_{\nu n}) = 0 \quad \Leftarrow \quad x_{\nu n}$ is the n^{th} root $\quad \Rightarrow$

$$\begin{aligned} \nu = 0,\quad & x_{0n} = 2.405,\quad 5.520,\quad 8.654,\cdots \\ \nu = 1,\quad & x_{1n} = 3.832,\quad 7.016,\quad 10.173,\cdots \\ \nu = 2,\quad & x_{2n} = 5.136,\quad 8.417,\quad 11.620,\cdots \end{aligned}$$

  $x_{\nu n} \simeq \dfrac{\pi}{4}(4\,n + 2\,\nu - 1) \quad \Leftarrow \quad$ asymptotic formula

- $\sqrt{\rho}\, J_\nu(x_{\nu n}\frac{\rho}{a})$ for fixed $\nu \ge 0$, $n = 1, 2, \cdots$ form an orthogonal set on $0 \le \rho \le a$

proof: $\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{d J_\nu(x_{\nu n}\rho/a)}{d\rho}\right] + (\frac{x_{\nu n}^2}{a^2} - \frac{\nu^2}{\rho^2}) J_\nu(x_{\nu n}\frac{\rho}{a}) = 0$

$$\Rightarrow \int_0^a J_\nu(x_{\nu n'}\frac{\rho}{a})\left\{\frac{d}{d\rho}\left[\rho\frac{d J_\nu(x_{\nu n}\rho/a)}{d\rho}\right] + (\frac{x_{\nu n}^2}{a^2} - \frac{\nu^2}{\rho^2})\rho J_\nu(x_{\nu n}\frac{\rho}{a})\right\} d\rho = 0$$

$$\Rightarrow \rho J_\nu(x_{\nu n'}\frac{\rho}{a})\frac{d J_\nu(x_{\nu n}\rho/a)}{d\rho}\Bigg|_0^a \qquad [= x_{\nu n} J_\nu(x_{\nu n'}) J'_\nu(x_{\nu n})]$$

$$+\int_0^a \rho\left[-\frac{d J_\nu(x_{\nu n'}\rho/a)}{d\rho}\frac{d J_\nu(x_{\nu n}\rho/a)}{d\rho} + (\frac{x_{\nu n}^2}{a^2} - \frac{\nu^2}{\rho^2}) J_\nu(x_{\nu n'}\frac{\rho}{a}) J_\nu(x_{\nu n}\frac{\rho}{a})\right] d\rho = 0 \quad (*')$$

write down the same equation, with n and n' interchanged, and sbutract

$$\Rightarrow (x_{\nu n}^2 - x_{\nu n'}^2)\int_0^a \rho J_\nu(x_{\nu n'}\frac{\rho}{a}) J_\nu(x_{\nu n}\frac{\rho}{a})\, d\rho = 0 \Leftarrow \text{ orthogonality condition} \quad \text{QED}$$

- The normalization integral: $\int_0^a \rho J_\nu(x_{\nu n'}\frac{\rho}{a}) J_\nu(x_{\nu n}\frac{\rho}{a})\, d\rho = \frac{a^2}{2} J_{\nu\pm1}^2(x_{\nu n})\,\delta_{n'n}$

Proof: (#) $\Rightarrow J_{\nu\pm1} = \frac{\nu}{x} J_\nu \mp J'_\nu \Rightarrow J_{\nu\pm1}(x_{\nu n}) = \mp J'_\nu(x_{\nu n}) \Rightarrow J'^2_\nu(x_{\nu n}) = J^2_{\nu\pm1}(x_{\nu n})$

$$(*')-(n\leftrightarrow n') \quad \Rightarrow \quad x_{\nu n} J_\nu(x_{\nu n'}) J'_\nu(x_{\nu n}) - x_{\nu n'} J_\nu(x_{\nu n}) J'_\nu(x_{\nu n'})$$
$$= \frac{x_{\nu n'}^2 - x_{\nu n}^2}{a^2} \int_0^a \rho J_\nu(x_{\nu n'} \frac{\rho}{a}) J_\nu(x_{\nu n} \frac{\rho}{a}) \, \mathrm{d}\rho$$

$$x_{\nu n'} = x_{\nu n} + \epsilon \quad \Rightarrow \quad J_\nu(x_{\nu n'}) \approx J_\nu(x_{\nu n}) + \epsilon J'_\nu(x_{\nu n})$$

$$\Rightarrow \quad x_{\nu n} J_\nu(x_{\nu n'}) J'_\nu(x_{\nu n}) - x_{\nu n'} J_\nu(x_{\nu n}) J'_\nu(x_{\nu n'}) \approx \epsilon \, x_{\nu n} J'^2_\nu(x_{\nu n}) \quad \Leftarrow \quad J_\nu(x_{\nu n}) = 0$$

$$\frac{x_{\nu n'}^2 - x_{\nu n}^2}{a^2} \int_0^a \rho J_\nu(x_{\nu n'} \frac{\rho}{a}) J_\nu(x_{\nu n} \frac{\rho}{a}) \, \mathrm{d}\rho \approx \frac{2 x_{\nu n} \epsilon}{a^2} \int_0^a \rho J_\nu^2(x_{\nu n} \frac{\rho}{a}) \, \mathrm{d}\rho$$

$$\Rightarrow \quad \int_0^a \rho J_\nu^2(x_{\nu n} \frac{\rho}{a}) \, \mathrm{d}\rho = \frac{a^2}{2} J'^2_\nu(x_{\nu n}) = \frac{a^2}{2} J_{\nu \pm 1}^2(x_{\nu n}) \quad \text{QED}$$

- Assuming the set of Bessel functions is complete, the Fourier-Bessel series of f

$$f(\rho) = \sum_{n=1}^{\infty} A_{\nu n} J_\nu(x_{\nu n} \frac{\rho}{a}), \quad \begin{matrix} 0 \le \rho < a \\ \nu \ge -1 \end{matrix} \quad \Leftarrow \quad A_{\nu n} = \frac{2}{a^2 J_{\nu+1}^2(x_{\nu n})} \int_0^a \rho f(\rho) J_\nu(x_{\nu n} \frac{\rho}{a}) \, \mathrm{d}\rho$$

- the Fourier-Bessel series is appropriate to functions vanishing at $\rho=a$, ie homogeneous Dirichlet boundary conditions on a cylinder.

- an alternative expansion is in a series of functions $\sqrt{\rho} J_\nu(y_{\nu n} \rho / a)$ where $y_{\nu n}$ is the n^{th} root of $J'_\nu(x)=0$, and is useful for functions with vanishing slope at $\rho=a$.

- The reason is that, in proving the orthogonality of the functions, it needs

$$J_\nu(x_{\nu n}) \quad \text{or} \quad J'_\nu(y_{\nu n}) = 0 \quad \Rightarrow \quad x\, J'_\nu(x) + \lambda\, J_\nu(x) = 0 \quad \text{at the endpoints} \quad \Leftarrow \quad \lambda = \begin{bmatrix} x_{\nu n}/a \\ y_{\nu n}/a \end{bmatrix}$$

- Some of the other possibilities

$$\text{Neumann series: } \sum_{n=0}^{\infty} a_n\, J_{\nu+n}(z), \qquad \text{Schlomilch series: } \sum_{n=0}^{\infty} a_n\, J_\nu(n\, x)$$

$$\text{Kapteyn series: } \sum_{n=0}^{\infty} a_n\, J_{\nu+n}[(\nu+n)\, z] \quad \Leftarrow \quad \text{Kepler motion of planets and of radiation by moving charges}$$

- If $\dfrac{\mathrm{d}^2 Z}{\mathrm{d}\, z^2} + k^2 Z = 0 \quad \Rightarrow \quad Z(z) = \sin k\, z\,,\ \cos k\, z$

$$\Rightarrow \quad \frac{\mathrm{d}^2 R}{\mathrm{d}\,\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}\, R}{\mathrm{d}\,\rho} - (k^2 + \frac{\nu^2}{\rho^2})\, R = 0 \quad \Rightarrow \quad \frac{\mathrm{d}^2 R}{\mathrm{d}\, x^2} + \frac{1}{x}\frac{\mathrm{d}\, R}{\mathrm{d}\, x} - (1 + \frac{\nu^2}{x^2})\, R = 0 \quad \Leftarrow \quad x = k\,\rho$$

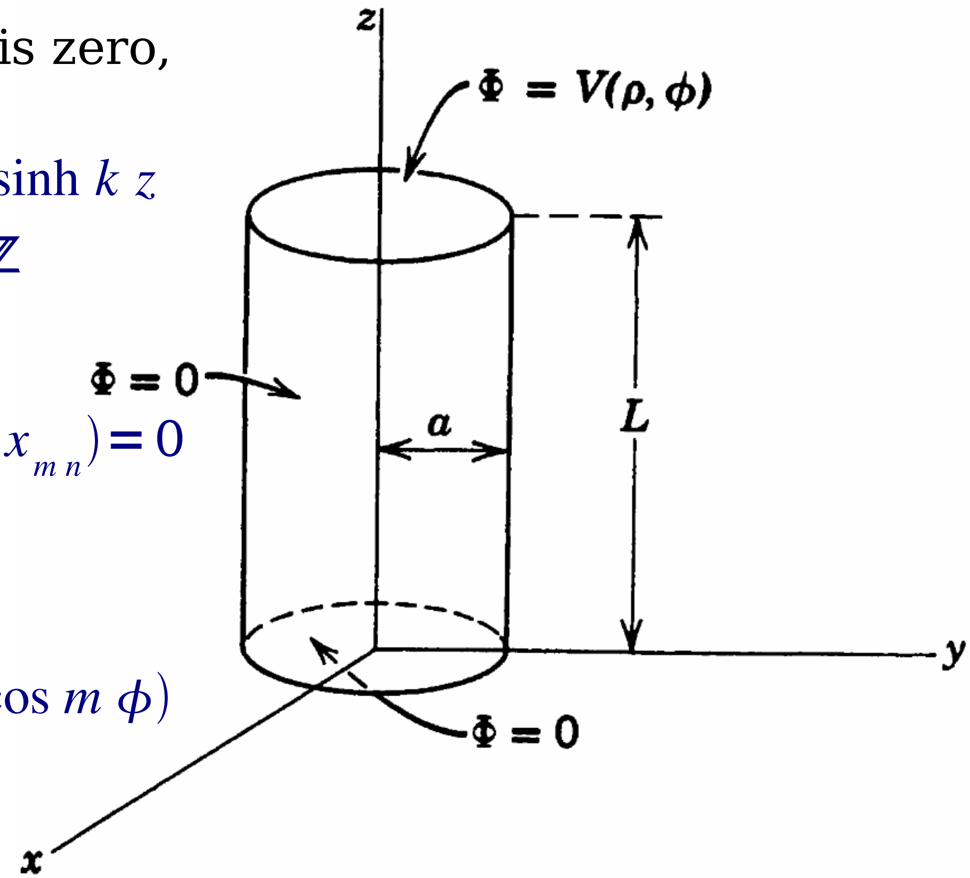
$$\Rightarrow \quad \boxed{I_\nu(x) = i^{-\nu} J_\nu(i\, x)\,, \qquad K_\nu(x) = \frac{\pi}{2}\, i^{\nu+1} H_\nu^{(1)}(i\, x)} \quad \Leftarrow \quad \text{modified Bessel functions}$$

$$\nu \geq 0 \quad \Rightarrow \quad x \ll 1 \quad \Rightarrow \quad I_\nu(x) \to \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu, \quad K_\nu(x) \to \begin{cases} -[\ln \dfrac{x}{2} + 0.5772\cdots]\,, & \nu = 0 \\ \dfrac{\Gamma(\nu)}{2}\left(\dfrac{2}{x}\right)^\nu, & \nu \neq 0 \end{cases}$$

$$x \gg 1, \nu \quad \Rightarrow \quad I_\nu(x) \to \frac{e^x}{\sqrt{2\,\pi\, x}}[1 + O(x^{-1})]\,, \quad K_\nu(x) \to \sqrt{\frac{\pi}{2\, x}}\, e^{-x}[1 + O(x^{-1})]$$

§3.8 Boundary-Value Problems in Cylindrical Coordinates

- The potential on the side and the bottom is zero, the top is $\Phi = V(\rho, \phi)$



$$\Rightarrow \quad Q(\phi) = A \sin m\phi + B \cos m\phi, \qquad Z(z) = \sinh kz$$

$$\Rightarrow \quad R(\rho) = C J_m(k\rho) + D N_m(k\rho) \quad \Leftarrow \quad m \in \mathbb{Z}$$

$$R(0) = \text{finite} \quad \Rightarrow \quad D = 0 \quad \Leftarrow \quad N_m(\rho = 0) \to \infty$$

$$R(ka) = 0 \quad \Rightarrow \quad k_{mn} = x_{mn}/a, \quad n \in \mathbb{N} \quad \Leftarrow \quad J_m(x_{mn}) = 0$$

$$\Rightarrow \quad \Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}\rho) \sinh k_{mn} z \, (A_{mn} \sin m\phi + B_{mn} \cos m\phi)$$

$$\Rightarrow \quad V = \sum_{m,n} J_m(k_{mn}\rho) \sinh k_{mn} L \, (A_{mn} \sin m\phi + B_{mn} \cos m\phi)$$

$$A_{mn} = \frac{2 \operatorname{csch} k_{mn} L}{\pi a^2 J_{m+1}^2(k_{mn} a)} \int_0^{2\pi} \mathrm{d}\phi \int_0^a \rho V J_m(k_{mn}\rho) \sin m\phi \, \mathrm{d}\rho$$

$$\Rightarrow$$

$$B_{mn} = \frac{2 \operatorname{csch} k_{mn} L}{\pi a^2 J_{m+1}^2(k_{mn} a)} \int_0^{2\pi} \mathrm{d}\phi \int_0^a \rho V J_m(k_{mn}\rho) \cos m\phi \, \mathrm{d}\rho$$

using $B_{0n}/2$ for $m = 0$

- If $\begin{array}{c} a \to \infty \\ 0 \le z \to \infty \end{array} \Rightarrow \Phi(\rho,\phi,z) = \sum_{m=0}^{\infty} \int_0^{\infty} e^{-kz} J_m(k\rho)\left[A_m(k)\sin m\phi + B_m(k)\cos m\phi\right] dk$

If $\Phi(z=0) = V \Rightarrow V(\rho,\phi) = \sum_{m=0}^{\infty} \int_0^{\infty} J_m(k\rho)\left[A_m(k)\sin m\phi + B_m(k)\cos m\phi\right] dk$

$$\Rightarrow \int_0^{\infty} J_m(k\rho)\left\{\begin{matrix} A_m(k) \\ B_m(k) \end{matrix}\right\} dk = \frac{1}{\pi}\int_0^{2\pi} V(\rho,\phi)\left\{\begin{matrix} \sin m\phi \\ \cos m\phi \end{matrix}\right\} d\phi$$

- These radial integral equations of the 1st kind can be solved since they are *Hankel transforms*.

- Using the integral relation $\int_0^{\infty} x J_m(kx) J_m(k'x)\, dx = \frac{1}{k}\delta(k'-k)$

$$\Rightarrow \left.\begin{matrix} A_m(k) \\ B_m(k) \end{matrix}\right\} = \frac{k}{\pi}\int_0^{\infty} \rho\, d\rho \int_0^{2\pi} d\phi\, V(\rho,\phi) J_m(k\rho)\left\{\begin{matrix} \sin m\phi \\ \cos m\phi \end{matrix}\right. \quad \text{\& using } \frac{1}{2}B_0(k) \text{ for } m=0$$

- $J_\nu(kx)$'s for fixed ν, Re(ν)>-1, form a complete, orthogonal (in k) set on the interval $0<x<\infty$. For m fixed

$$A(x) = \int_0^{\infty} \tilde{A}(k) J_\nu(kx)\, dk \quad \Leftarrow \quad \tilde{A}(k) = k\int_0^{\infty} x A(x) J_\nu(kx)\, dx$$

- Spherical Bessel function $j_\ell(z) \equiv \sqrt{\frac{\pi}{2z}}\, J_{\ell+1/2}(z), \quad z \in \mathbb{N} \cup 0$

$$\Rightarrow \quad \begin{aligned} &\int_0^\infty r^2\, j_\ell(k\, r)\, j_\ell(k'\, r)\, \mathrm{d}\, r = \frac{\pi}{2\,k^2}\, \delta(k-k') \quad \Leftarrow \quad \text{orthogonality relation} \\ &\int_0^\infty k^2\, j_\ell(k\, r)\, j_\ell(k\, r')\, \mathrm{d}\, k = \frac{\pi}{2\,r^2}\, \delta(r-r') \quad \Leftarrow \quad \text{completeness relation} \end{aligned}$$

- The Fourier-spherical Bessel expansion for a given ℓ

$$A(r) = \int_0^\infty \tilde{A}(k)\, j_\ell(k\, r)\, \mathrm{d}\, k \quad \Leftarrow \quad \tilde{A}(k) = \frac{2\,k^2}{\pi} \int_0^\infty r^2\, A(r)\, j_\ell(k\, r)\, \mathrm{d}\, r$$

- useful for current decay in conducting media or time-dependent magnetic diffusion with angular symmetry. [Chapter 9]

§3.9 Expansion of Green Functions in Spherical Coordinates

- To handle problems involving distributions of charge and boundary values for the solutions of the Poisson equation, it needs to determine the Green function that satisfies the appropriate boundary conditions.

- For the case of no boundary surfaces, the expansion of the Green function

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\frac{r_<^{\ell}}{r_>^{\ell+1}}Y_{\ell m}^{*}(\theta',\phi')Y_{\ell m}(\theta,\phi)$$

- to obtain a similar expansion for the Green function for the "exterior" problem with a spherical boundary at $r=a$, which is found from the image form method

$$G(\mathbf{x},\mathbf{x}')=\sum_{\ell,m}\frac{4\pi}{2\ell+1}\left[\frac{r_<^{\ell}}{r_>^{\ell+1}}-\frac{1}{a}\left(\frac{a^2}{r_<r_>}\right)^{\ell+1}\right]Y_{\ell m}^{*}(\theta',\phi')Y_{\ell m}(\theta,\phi)$$

where
$$\frac{r_<^{\ell}}{r_>^{\ell+1}}-\frac{1}{a}\left(\frac{a^2}{r_<r_>}\right)^{\ell+1}=\begin{cases}\dfrac{1}{r'^{\ell+1}}\left[r^{\ell}-\dfrac{a^{2\ell+1}}{r^{\ell+1}}\right], & r<r' \Rightarrow \text{vanishes for } r=a \text{ or } r'\to\infty \\[2ex] \dfrac{1}{r^{\ell+1}}\left[r'^{\ell}-\dfrac{a^{2\ell+1}}{r'^{\ell+1}}\right], & r>r' \Rightarrow \text{vanishes for } r'=a \text{ or } r\to\infty\end{cases}$$

- symmetric in r and r'.

- The reason for different linear combinations for $r<r'$ and for $r>r'$ is connected with the fact that the Green function is a solution of the Poisson equation with a delta function inhomogeneity.

- From first principles, a Green function for a Dirichlet potential problem

$$\nabla_x^2 G(\mathbf{x}, \mathbf{x'}) = -4\pi\,\delta(\mathbf{x}-\mathbf{x'}) \quad \Leftarrow \quad G(\mathbf{x}, \mathbf{x'}) = 0 \quad \text{for either } \mathbf{x} \text{ or } \mathbf{x'} \text{ on the boundary}$$

$$\delta(\mathbf{x}-\mathbf{x'}) = \frac{1}{r^2}\,\delta(r-r')\,\delta(\phi-\phi')\,\delta(\cos\theta-\cos\theta')$$

$$= \frac{1}{r^2}\,\delta(r-r')\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\phi')\,Y_{\ell m}(\theta,\phi)$$

$$\Rightarrow \quad G(\mathbf{x}, \mathbf{x'}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_{\ell m}(r|r',\theta',\phi')\,Y_{\ell m}(\theta,\phi)$$

$$\Rightarrow \quad \begin{bmatrix} A_{\ell m}(r|r',\theta',\phi') = g_\ell(r,r')\,Y^*_{\ell m}(\theta',\phi') \\ \dfrac{1}{r}\dfrac{\mathrm{d}^2}{\mathrm{d}r^2}\left[r\,g_\ell(r,r')\right] - \dfrac{\ell(\ell+1)}{r^2}\,g_\ell(r,r') = -\dfrac{4\pi}{r^2}\,\delta(r-r') \end{bmatrix}$$

$$\Rightarrow \quad g_\ell(r,r') = \begin{cases} A\ \ r^\ell + B\ \ r^{-(\ell+1)} & \text{for} \quad r<r' \\ A'\,r^\ell + B'\,r^{-(\ell+1)} & \text{for} \quad r>r' \end{cases}$$

- A, B, A', B' are functions of r' to be determined by the boundary conditions, the requirement implied by $\delta(r\text{-}r')$, and the symmetry of $g_\ell(r, r')$ in r and r'.

- Suppose that the boundary surfaces are concentric spheres at $r=a$ and $r=b$

$$\Rightarrow \quad g_\ell(a,r')=g_\ell(b,r')=0 \quad \Rightarrow \quad g_\ell(r,r')=\begin{cases} A\left(r^\ell - \frac{a^{2\ell+1}}{r^{\ell+1}}\right), & \text{for } r<r' \\ B'\left(\frac{1}{r^{\ell+1}} - \frac{r^\ell}{b^{2\ell+1}}\right), & \text{for } r>r' \end{cases}$$

$$\Rightarrow \quad g_\ell(r,r')=C\left(r_<^\ell - \frac{a^{2\ell+1}}{r_<^{\ell+1}}\right)\left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}}\right) \quad \Leftarrow \quad \text{symmetry in } r \text{ and } r'$$

$$\int_{r'-\epsilon}^{r'+\epsilon}\left[\frac{d^2}{dr^2}[r\,g_\ell(r,r')] - \frac{\ell(\ell+1)}{r}g_\ell(r,r')\right]dr = -\int_{r'-\epsilon}^{r'+\epsilon}\frac{4\pi}{r}\delta(r-r')\,dr$$

$$\Rightarrow \quad \frac{d}{dr}[r\,g_\ell(r,r')]\Big|_{r'+\epsilon} - \frac{d}{dr}[r\,g_\ell(r,r')]\Big|_{r'-\epsilon} = -\frac{4\pi}{r'}$$

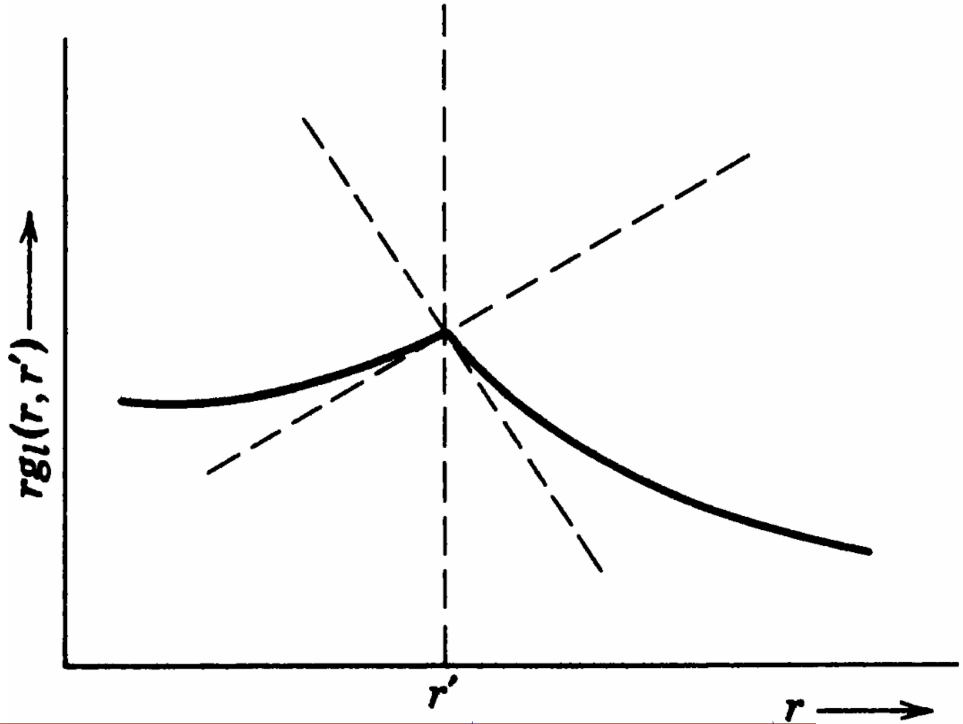
$$\frac{d}{dr}[r\,g_\ell(r,r')]\Big|_{r'+\epsilon} = C\left(r'^\ell - \frac{a^{2\ell+1}}{r'^{\ell+1}}\right)\left[\frac{d}{dr}\left(\frac{1}{r^\ell} - \frac{r^{\ell+1}}{b^{2\ell+1}}\right)\right]_{r=r'}$$

$$= -\frac{C}{r'}\left[1-\left(\frac{a}{r'}\right)^{2\ell+1}\right]\left[\ell+(\ell+1)\left(\frac{r'}{b}\right)^{2\ell+1}\right]$$

$$\frac{d}{dr}[r\,g_\ell(r,r')]\Big|_{r'-\epsilon}$$

$$=\frac{C}{r'}\left[1-\left(\frac{r'}{b}\right)^{2\ell+1}\right]\left[\ell+1+\ell\left(\frac{a}{r'}\right)^{2\ell+1}\right]$$

$$\Rightarrow \quad C=\frac{4\pi}{(2\ell+1)\left[1-\left(\frac{a}{b}\right)^{2\ell+1}\right]}$$



$$\Rightarrow \quad G(\mathbf{x},\mathbf{x'})=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi\,Y^*_{\ell m}(\theta',\phi')\,Y_{\ell m}(\theta,\phi)}{(2\ell+1)[1-(a/b)^{2\ell+1}]}\left(r_<^\ell-\frac{a^{2\ell+1}}{r_<^{\ell+1}}\right)\left(\frac{1}{r_>^{\ell+1}}-\frac{r_>^\ell}{b^{2\ell+1}}\right)$$

$$\to\sum_{\ell,m}\frac{4\pi}{2\ell+1}\left[\frac{r_<^\ell}{r_>^{\ell+1}}-\frac{1}{a}\left(\frac{a^2}{r_<r_>}\right)^{\ell+1}\right]Y^*_{\ell m}(\theta',\phi')\,Y_{\ell m}(\theta,\phi)\quad \text{for}\quad b\to\infty$$

$$\to\sum_{\ell,m}\frac{4\pi}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y^*_{\ell m}(\theta',\phi')\,Y_{\ell m}(\theta,\phi)\qquad \text{for}\quad a\to 0,\ \ b\to\infty$$

$$\to\sum_{\ell,m}\frac{4\pi}{2\ell+1}\left[\frac{r_<^\ell}{r_>^{\ell+1}}-\frac{1}{b}\left(\frac{r_<r_>}{b^2}\right)^{\ell}\right]Y^*_{\ell m}(\theta',\phi')\,Y_{\ell m}(\theta,\phi)\qquad \text{for}\quad a\to 0$$

§3.10 Solution of Potential Problems with the Spherical Green Function Expansion

- The general solution to the Poisson eqn with specified values of the potential on the boundary surface

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x'})\, G(\mathbf{x},\mathbf{x'})\, d^3x' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{x'})\frac{\partial G}{\partial n'}\, da'$$

- consider the potential *inside* a sphere of radius b and set a=0

$$\frac{\partial G}{\partial n'} = \left.\frac{\partial G}{\partial r'}\right|_{r'=b} = -\frac{4\pi}{b^2}\sum_{\ell,m}\left(\frac{r}{b}\right)^{\ell} Y^*_{\ell m}(\theta',\phi')\, Y_{\ell m}(\theta,\phi) \quad \text{and} \quad \begin{matrix}\Phi(\mathbf{x'}) = V(\theta,\phi)\\ \rho(\mathbf{x'}) = 0\end{matrix}$$

$$\Rightarrow \quad \Phi(\mathbf{x}) = \sum_{\ell m}\left[\int V(\theta',\phi')\, Y^*_{\ell m}(\theta',\phi')\, d\Omega'\right]\left(\frac{r}{b}\right)^{\ell} Y_{\ell m}(\theta,\phi)$$

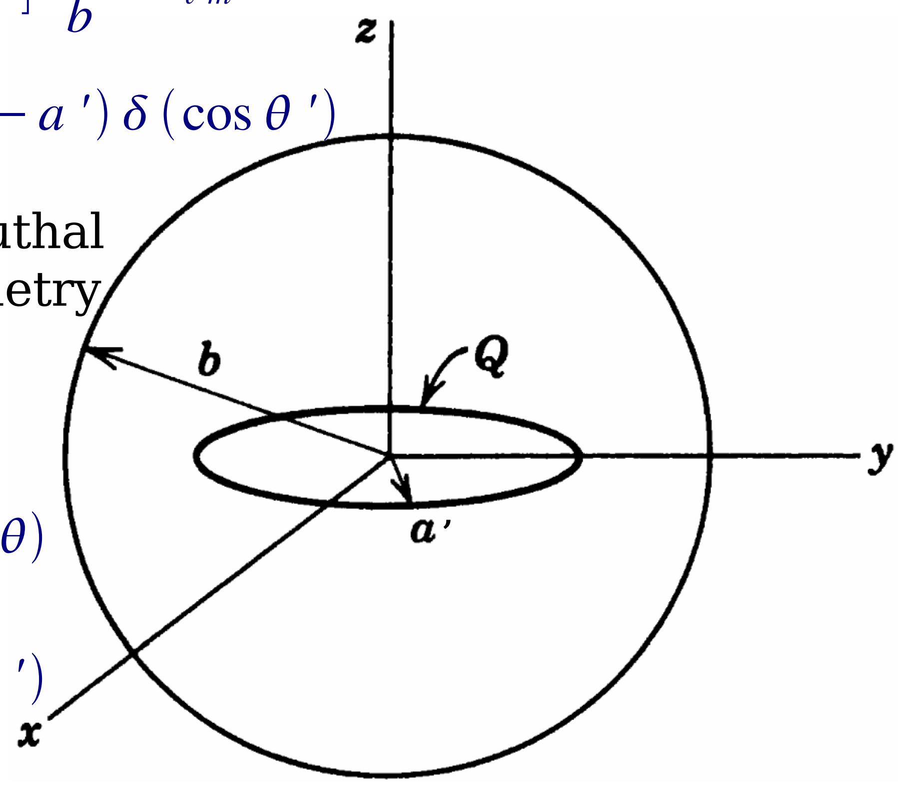
- Now consider $V=0 \quad \& \quad \rho(\mathbf{x'}) = \frac{Q}{2\pi a'^2}\delta(r'-a')\,\delta(\cos\theta')$

only terms with m=0 survive because of azimuthal symmetry

$$\Rightarrow \Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{x'})\, G(\mathbf{x},\mathbf{x'})\, d^3x'$$

$$= \frac{Q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty} P_\ell(0)\left(\frac{r_<^{\ell}}{r_>^{\ell+1}} - \frac{r_<^{\ell} r_>^{\ell}}{b^{2\ell+1}}\right) P_\ell(\cos\theta)$$

where $\quad r_< = \min(r, a'), \quad r_> = \max(r, a')$



$$P_{2n+1}(0)=0, \quad P_{2n}(0)=\frac{(-1)^n(2n-1)!!}{2^n n!}$$

$$\Rightarrow \quad \Phi(\mathbf{x})=\frac{Q}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n n!}\left(\frac{r_<^{2n}}{r_>^{2n+1}}-\frac{r_<^{2n}r_>^{2n}}{b^{4n+1}}\right)P_{2n}(\cos\theta)$$

$$\to \frac{Q}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n n!}\frac{r_<^{2n}}{r_>^{2n+1}}P_{2n}(\cos\theta) \quad \Leftarrow \quad b\to\infty \quad \Rightarrow \quad \text{Sec. III}$$

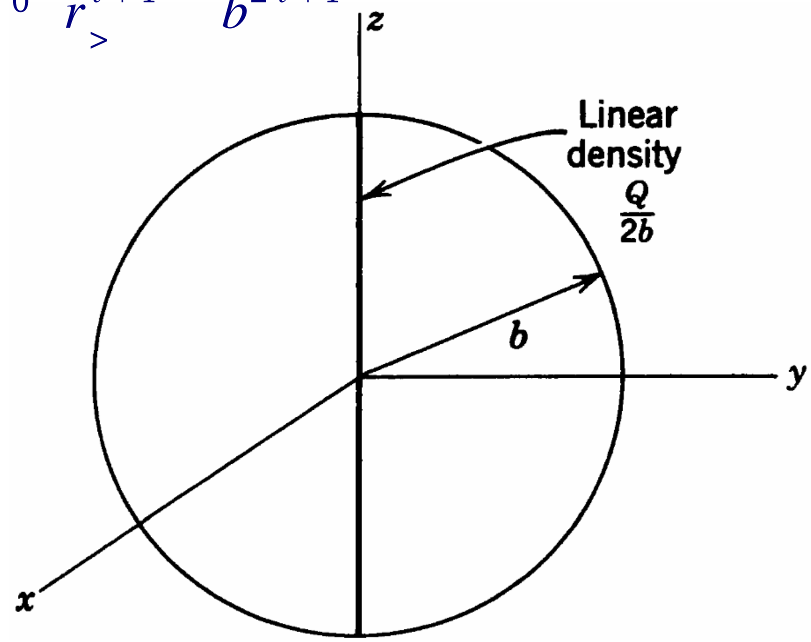
- 2^{nd} example: $V=0 \quad \& \quad \rho(\mathbf{x}')=\dfrac{Q}{4\pi b r'^2}[\delta(\cos\theta'-1)+\delta(\cos\theta'+1)]$

$$\Rightarrow \Phi(\mathbf{x})=\frac{Q}{8\pi\epsilon_0 b}\sum_{\ell=0}^{\infty}[P_\ell(1)+P_\ell(-1)]P_\ell(\cos\theta)\int_0^b\left(\frac{r_<^\ell}{r_>^{\ell+1}}-\frac{r_<^\ell r_>^\ell}{b^{2\ell+1}}\right)dr'$$

$$\int_0^b\left(\frac{r_<^\ell}{r_>^{\ell+1}}-\frac{r_<^\ell r_>^\ell}{b^{2\ell+1}}\right)dr'=\left(\frac{1}{r^{\ell+1}}-\frac{r^\ell}{b^{2\ell+1}}\right)\int_0^r r'^\ell\, dr'$$

$$+r^\ell\int_r^b\left(\frac{1}{r'^{\ell+1}}-\frac{r'^\ell}{b^{2\ell+1}}\right)dr'$$

$$=\frac{2\ell+1}{\ell(\ell+1)}\left[1-\left(\frac{r}{b}\right)^\ell\right]$$



special for $\ell=0$: $\int_0^b \left(\frac{1}{r_>}-\frac{1}{b}\right) d r' = \left(\frac{1}{r}-\frac{1}{b}\right)\int_0^r d r' + \int_r^b \left(\frac{1}{r'}-\frac{1}{b}\right) d r' = \ln\frac{b}{r}$

$$\Rightarrow \quad \Phi(\mathbf{x}) = \frac{Q}{4\pi\epsilon_0 b}\left\{\ln\frac{b}{r} + \sum_{\ell=1}^{\infty}\frac{4\ell+1}{2\ell(2\ell+1)}\left[1-\left(\frac{r}{b}\right)^{2\ell}\right]P_{2\ell}(\cos\theta)\right\} \quad \Leftarrow \quad P_\ell(-1)=(-1)^\ell$$

- the potential diverges along the z axis for $\cos\theta=\pm1$, except at $r=b$ exactly.

- The surface-charge density on the grounded sphere

$$\sigma(\theta) = \epsilon_0 \left.\frac{\partial\Phi}{\partial r}\right|_{r=b} = -\frac{Q}{4\pi b^2}\left[1+\sum_{\ell=1}^{\infty}\frac{4\ell+1}{2\ell+1}P_{2\ell}(\cos\theta)\right]$$

- The leading term shows that the total charge induced on the sphere is -Q, the other terms integrating to zero over the surface of the sphere.

§3.11 Expansion of Green Functions in Cylindrical Coordinates

- the eqn for the Green function: $\nabla_x^2 G(\mathbf{x},\mathbf{x}') = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')$
- The delta functions can be written in terms of orthonormal functions:

$$\delta(\phi-\phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}$$

$$\delta(z-z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(z-z')}\,\mathrm{d}k = \frac{1}{\pi}\int_0^{\infty}\cos k(z-z')\,\mathrm{d}k$$

$$\Rightarrow \quad G(\mathbf{x},\mathbf{x}') = \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{\infty} e^{im(\phi-\phi')}\cos k(z-z')\, g_m(k,\rho,\rho')\,\mathrm{d}k$$

$$\Rightarrow \quad \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}g_m}{\mathrm{d}\rho}\right) - \left(k^2+\frac{m^2}{\rho^2}\right)g_m = -\frac{4\pi}{\rho}\delta(\rho-\rho') \quad \Rightarrow \quad \begin{matrix} I_m(k\rho) \\ K_m(k\rho) \end{matrix} \quad \text{for} \quad \rho\neq\rho'$$

Let
$$\begin{cases} \psi_1(k\rho) = A\,I_m(k\rho) + B\,K_m(k\rho) & \text{satifies the boundary condition for } \rho<\rho' \\ \psi_2(k\rho) = C\,I_m(k\rho) + D\,K_m(k\rho) & \text{satifies the boundary condition for } \rho>\rho' \end{cases}$$

$$\Rightarrow \quad g_m(k,\rho,\rho') = \psi_1(k\rho_<)\,\psi_2(k\rho_>) \quad \Leftarrow \quad \text{the symmetry of the Green function}$$

$$\Rightarrow \quad \left.\frac{\mathrm{d}g_m}{\mathrm{d}\rho}\right|_{\rho'+\epsilon} - \left.\frac{\mathrm{d}g_m}{\mathrm{d}\rho}\right|_{\rho'-\epsilon} = k\begin{vmatrix} \psi_1 & \psi_2 \\ \psi_1' & \psi_2' \end{vmatrix}\left(= k\,\underbrace{W[\psi_1,\psi_2]}_{\text{Wronskian}}\right) = -\frac{4\pi}{\rho'}$$

- the Sturm-Liouville type equation: $\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+g(x)\,y=0$
- the Wronskian of 2 linearly independent solutions of a Sturm-Liouville type eqn is proportional to $1/p(x)$ $\Rightarrow$ $W[\psi_1(x),\psi_2(x)]=-\frac{4\pi}{x}$
- $\begin{bmatrix} g_m(k,0,\rho') \text{ is finite} \\ g_m(k,\infty,\rho')=0 \end{bmatrix} \Rightarrow \begin{bmatrix} \psi_1(k\rho)=A\,I_m(k\rho) \\ \psi_2(k\rho)=K_m(k\rho) \end{bmatrix} \Rightarrow A=4\pi \Leftarrow W[I_m(x),K_m(x)]=-\frac{1}{x}$

$$\Rightarrow \frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} I_m(k\rho_<)\,K_m(k\rho_>)\cos k(z-z')\,e^{im(\phi-\phi')}\,dk$$

$$=\frac{4}{\pi}\int_0^{\infty}\left[\frac{1}{2}I_0(k\rho_<)\,K_0(k\rho_>)+\sum_{m=1}^{\infty}I_m(k\rho_<)\,K_m(k\rho_>)\cos m(\phi-\phi')\right]\cos k(z-z')\,dk$$

- $\mathbf{x}'\to 0$ $\Rightarrow$ only the $m=0$ term survives $\Rightarrow$ $\frac{1}{\sqrt{\rho^2+z^2}}=\frac{2}{\pi}\int_0^{\infty}K_0(k\rho)\cos kz\,dk$

$$\Rightarrow \quad K_0\left(k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}\right)$$

$$=I_0(k\rho_<)\,K_0(k\rho_>)+2\sum_{m=1}^{\infty}I_m(k\rho_<)\,K_m(k\rho_>)\cos m(\phi-\phi')$$

$$\Rightarrow \quad \ln\frac{1}{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}=2\ln\frac{1}{\rho_<}+2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos m(\phi-\phi') \quad \Leftarrow k\to 0$$

§3.12 Eigenfunction Expansions for Green Functions

- Elliptic differential equation $\nabla^2 \psi(\mathbf{x}) + [f(\mathbf{x}) + \lambda]\psi(\mathbf{x}) = 0$

- If the solutions are required to satisfy homogeneous boundary conditions on the surface S of the volume of V, then the eqn will not in general have well-behaved solutions, except for certain values of λ.

- These values of λ, denoted by λ_n, are called *eigenvalues* (or *characteristic values*) and the solutions $\psi_n(x)$ are called *eigenfunctions* : $\nabla^2 \psi_n(\mathbf{x}) + [f(\mathbf{x}) + \lambda_n]\psi_n(\mathbf{x}) = 0$

- the eigenfunctions are orthogonal: $\int_V \psi_m^*(\mathbf{x})\,\psi_n(\mathbf{x})\,\mathrm{d}^3x = \delta_{mn}$ & assumed complete

- The spectrum of eigenvalues λ_n may be a discrete set, or a continuum, or both.

- To find the Green function: $\nabla_x^2 G(\mathbf{x},\mathbf{x}') + [f(\mathbf{x}) + \lambda]G(\mathbf{x},\mathbf{x}') = -4\pi\,\delta(\mathbf{x}-\mathbf{x}')$

$$\Rightarrow \quad G(\mathbf{x},\mathbf{x}') = \sum_n a_n(\mathbf{x}')\,\psi_n(\mathbf{x}) \quad \Rightarrow \quad \sum_m a_m(\mathbf{x}')(\lambda-\lambda_m)\,\psi_m(\mathbf{x}) = -4\pi\,\delta(\mathbf{x}-\mathbf{x}')$$

$$\Rightarrow \quad a_n(\mathbf{x}') = 4\pi\frac{\psi_n^*(\mathbf{x}')}{\lambda_n-\lambda} \quad \Leftarrow \quad \int_V \psi_n^*(\mathbf{x})\Big[\sum_m a_m(\mathbf{x}')(\lambda-\lambda_m)\,\psi_m(\mathbf{x}) = -4\pi\,\delta(\mathbf{x}-\mathbf{x}')\Big]\mathrm{d}^3x$$

$$\Rightarrow \quad \boxed{G(\mathbf{x},\mathbf{x}') = 4\pi\sum_n \frac{\psi_n^*(\mathbf{x}')\,\psi_n(\mathbf{x})}{\lambda_n-\lambda} \;\to\; 4\pi\int \frac{\psi_n^*(\mathbf{x}',\lambda')\,\psi_n(\mathbf{x},\lambda')}{\lambda'-\lambda}\,\mathrm{d}\lambda' \;\Leftarrow\; \text{for continuous}}$$

- Consider $f(\mathbf{x}) = \lambda = 0 \quad \Rightarrow \quad \nabla_x^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\,\delta(\mathbf{x} - \mathbf{x}')$
- the eigenfunctions for the wave equation: $(\nabla^2 + k^2)\,\psi_{\mathbf{k}}(\mathbf{x}) = 0 \quad \Rightarrow \quad \psi_{\mathbf{k}}(\mathbf{x}) = \dfrac{e^{i\,\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}$

$$\Rightarrow \quad \int \psi^*_{\mathbf{k}'}(\mathbf{x})\,\psi_{\mathbf{k}}(\mathbf{x})\,\mathrm{d}^3x = \delta(\mathbf{k} - \mathbf{k}') \quad \Rightarrow \quad G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} = \underbrace{\frac{1}{2\pi^2} \int \frac{e^{i\,\mathbf{k}\cdot(\mathbf{x} - \mathbf{x}')}}{k^2}\,\mathrm{d}^3k}_{\text{3d Fourier integration of } 1/|\mathbf{x}-\mathbf{x}'|}$$

- 2nd ex: a box defined with $x=y=z=0$, $x=a$, $y=b$, $z=c$: $(\nabla^2 + k^2_{\ell m n})\,\psi_{\ell m n}(x, y, z) = 0$

$$\Rightarrow \quad \psi_{\ell m n}(x, y, z) = \sqrt{\frac{8}{a\,b\,c}}\,\sin\frac{\ell\pi x}{a}\,\sin\frac{m\pi y}{b}\,\sin\frac{n\pi z}{c} \quad \Leftarrow \quad k^2_{\ell m n} = \pi^2\left(\frac{\ell^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)$$

$$\Rightarrow \quad G(\mathbf{x}, \mathbf{x}') = \frac{32}{\pi\,a\,b\,c} \sum_{\ell, m, n = 1}^{\infty} \frac{\sin\frac{\ell\pi x}{a}\,\sin\frac{\ell\pi x'}{a}\,\sin\frac{m\pi y}{b}\,\sin\frac{m\pi y'}{b}\,\sin\frac{n\pi z}{c}\,\sin\frac{n\pi z'}{c}}{\ell^2/a^2 + m^2/b^2 + n^2/c^2}$$

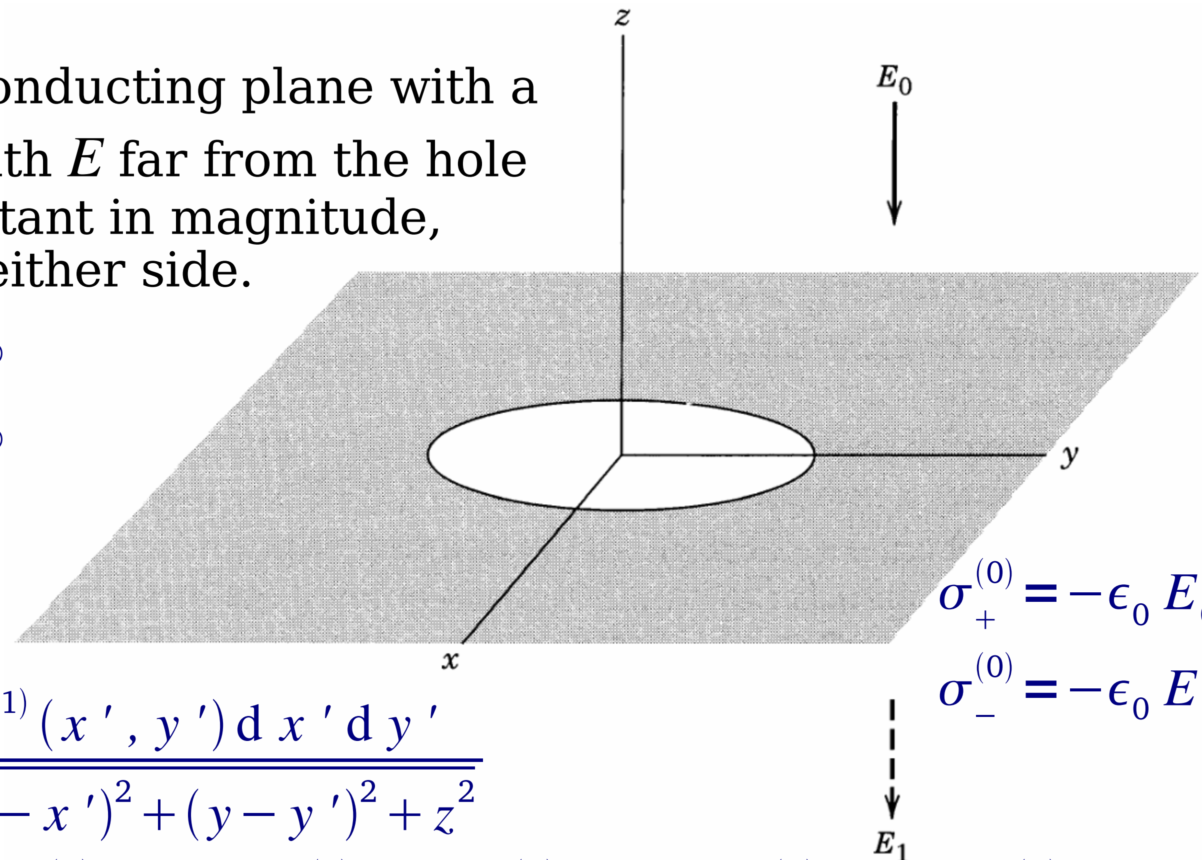
$$= \frac{16\pi}{a\,b} \sum_{\ell, m = 1}^{\infty} \sin\frac{\ell\pi x}{a}\,\sin\frac{\ell\pi x'}{a}\,\sin\frac{m\pi y}{b}\,\sin\frac{m\pi y'}{b}\,\frac{\sinh(K_{\ell m} z_<)\,\sinh[K_{\ell m}(c - z_>)]}{K_{\ell m}\,\sinh(K_{\ell m} c)}$$

$$\Rightarrow \quad \frac{\sinh(K_{\ell m} z_<)\,\sinh[K_{\ell m}(c - z_>)]}{K_{\ell m}\,\sinh(K_{\ell m} c)} = \frac{2}{c} \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi z'}{c}}{K^2_{\ell m} + \left(\frac{n\pi}{c}\right)^2}\,\sin\frac{n\pi z}{c} \quad \Leftarrow \quad K_{\ell m} = \pi\sqrt{\frac{\ell^2}{a^2} + \frac{m^2}{b^2}}$$

§3.13 Mixed Boundary Conditions; Conducting Plane with a Circular Hole

- Mixed boundary conditions are more difficult to handle than the normal one.

- an infinitely thin, grounded, conducting plane with a circular hole of radius a, and with E far from the hole being normal to the plane, constant in magnitude, and having different values on either side.



$$E = \begin{cases} -E_0 & \text{for} \quad z \to +\infty \\ -E_1 & \text{for} \quad z \to -\infty \end{cases}$$

$$\sigma_+^{(0)} = -\epsilon_0 E_0$$

$$\sigma_-^{(0)} = -\epsilon_0 E_1$$

$$\Rightarrow \quad \Phi = \begin{cases} E_0 z + \Phi^{(1)} & \text{for} \quad z > 0 \\ E_1 z + \Phi^{(1)} & \text{for} \quad z < 0 \end{cases}$$

$$\Rightarrow \quad \Phi^{(1)}(x, y, z) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma^{(1)}(x', y')\, dx'\, dy'}{\sqrt{(x-x')^2 + (y-y')^2 + z^2}}$$

$$\Phi^{(1)}(z) = \Phi^{(1)}(-z) \quad \Rightarrow \quad E_x^{(1)}(z) = E_x^{(1)}(-z), \quad E_y^{(1)}(z) = E_y^{(1)}(-z), \quad E_z^{(1)}(z) = -E_z^{(1)}(-z)$$

- the *total z-component* of E field must be continuous across z=0 in the hole

$$-E_0 + E_z^{(1)}\Big|_{z=0^+} = -E_1 + E_z^{(1)}\Big|_{z=0^-} \quad \text{for} \quad \rho < a \quad \Rightarrow \quad E_z^{(1)}\Big|_{z=0^+} = -E_z^{(1)}\Big|_{z=0^-} = \frac{E_0 - E_1}{2}$$

- the potential is zero on the surface by hypothesis: $\Phi(a \le \rho < \infty, z = 0) = 0$

- an electrostatic boundary-value problem with the *mixed boundary conditions*:

$$\left.\frac{\partial \Phi^{(1)}}{\partial z}\right|_{z=0^+} = \frac{E_1 - E_0}{2} \quad \text{for} \quad 0 \le \rho < a \qquad (\&) \quad \Rightarrow \quad \Phi^{(1)}(\rho, z) = \int_0^\infty A(k)\, J_0(k\rho)\, e^{-k|z|}\, \mathrm{d}k$$

$$\left.\Phi^{(1)}\right|_{z=0} = 0 \qquad \text{for} \quad a \le \rho < \infty$$

- For large ρ or $|z|$ the rapid oscillations of $J_0(k\rho)$ or the decrease of $e^{-k|z|}$ imply that the integral receives its important contributions from the region around $k=0$

$$A(k) = \sum_{\ell=0}^{\infty} \frac{k^\ell}{\ell!} \frac{\mathrm{d}^\ell A}{\mathrm{d}k^\ell}(0) \quad \Rightarrow \quad \Phi^{(1)} = \sum_{\ell=0}^{\infty} \frac{\mathrm{d}^\ell A}{\mathrm{d}k^\ell}(0)\, B_\ell \quad \Leftarrow \quad B_\ell = \frac{1}{\ell!} \int_0^\infty k^\ell J_0(k\rho)\, e^{-k|z|}\, \mathrm{d}k$$

$$\Rightarrow \quad B_\ell = \frac{(-1)^\ell}{\ell!} \frac{\mathrm{d}^\ell}{\mathrm{d}|z|^\ell} \int_0^\infty J_0(k\rho)\, e^{-k|z|}\, \mathrm{d}k = \frac{(-1)^\ell}{\ell!} \frac{\mathrm{d}^\ell}{\mathrm{d}|z|^\ell} \frac{1}{\sqrt{\rho^2 + z^2}} = \frac{P_\ell(|\cos\theta|)}{r^{\ell+1}}$$

$$\Rightarrow \quad \Phi^{(1)} = \sum_{\ell=0}^{\infty} \frac{P_\ell(\cos\theta)}{r^{\ell+1}} \frac{\mathrm{d}^\ell A}{\mathrm{d}k^\ell}(0) \quad \Leftarrow \quad \text{multipole expansion} \quad \begin{cases} A(0) & \text{total charge} \\ \dfrac{\mathrm{d}A}{\mathrm{d}k}(0) & \text{dipole moment} \\ \cdots & \end{cases}$$

- For the mixed boundary value problem

$$(\&) \quad \Rightarrow \quad \begin{aligned} &\int_0^\infty k\, A(k)\, J_0(k\rho)\, \mathrm{d}k = \frac{E_0 - E_1}{2} && \text{for} \quad 0 \le \rho < a \\ &\int_0^\infty A(k)\, J_0(k\rho)\, \mathrm{d}k = 0 && \text{for} \quad a \le \rho < \infty \end{aligned} \qquad \textit{dual integral equations}$$

- Consider the dual integral equations

$$\begin{aligned}&\int_0^\infty y\, g(y)\, J_n(y\,x)\, \mathrm{d}\,y = x^n \quad \text{for} \quad 0 \le x < 1\\ &\int_0^\infty g(y)\, J_n(y\,x)\, \mathrm{d}\,y = 0 \quad \text{for} \quad 1 \le x < \infty\end{aligned} \Rightarrow \quad g(y) = \frac{\Gamma(n+1)\, j_{n+1}(y)}{\Gamma(n+3/2)\sqrt{\pi}} = \frac{\Gamma(n+1)\, J_{n+3/2}(y)}{\Gamma(n+3/2)\sqrt{2\,y}}$$

$$\begin{bmatrix} n=0 \\ x=\rho/a \\ y=k\,a \end{bmatrix} \quad \Rightarrow \quad A(k) = \frac{a^2}{\pi}(E_0 - E_1)\, j_1(k\,a) = \frac{E_0 - E_1}{\pi}\left[\frac{\sin k\,a}{k^2} - \frac{a\cos k\,a}{k}\right]$$

$$\Rightarrow \quad A(k \to 0) \simeq \frac{a^2 (E_0 - E_1)}{3\,\pi}\left[k\,a - \frac{(k\,a)^3}{10} + \cdots\right] \quad \Rightarrow \quad A(0) = 0, \quad \frac{\mathrm{d}\,A}{\mathrm{d}\,k}(0) \neq 0$$

- The total charge with $\Phi^{(1)}$ is zero and the leading term is the $\ell=1$ contribution.

$$\Phi^{(1)} \to \frac{a^3 (E_0 - E_1)}{3\,\pi}\frac{|z|}{r^3} \quad \Rightarrow \quad \mathbf{p} = \mp\frac{4}{3}\epsilon_0\, a^3 (\mathbf{E_0} - \mathbf{E_1}) \quad \text{for } z \gtrless 0 \quad \Leftarrow \quad \textit{effective electric dipole moment}$$

- The reversal of the effective dipole moment depending on whether the observer is above or below the plane is because that a true dipole potential is odd in z, whereas it is even here.

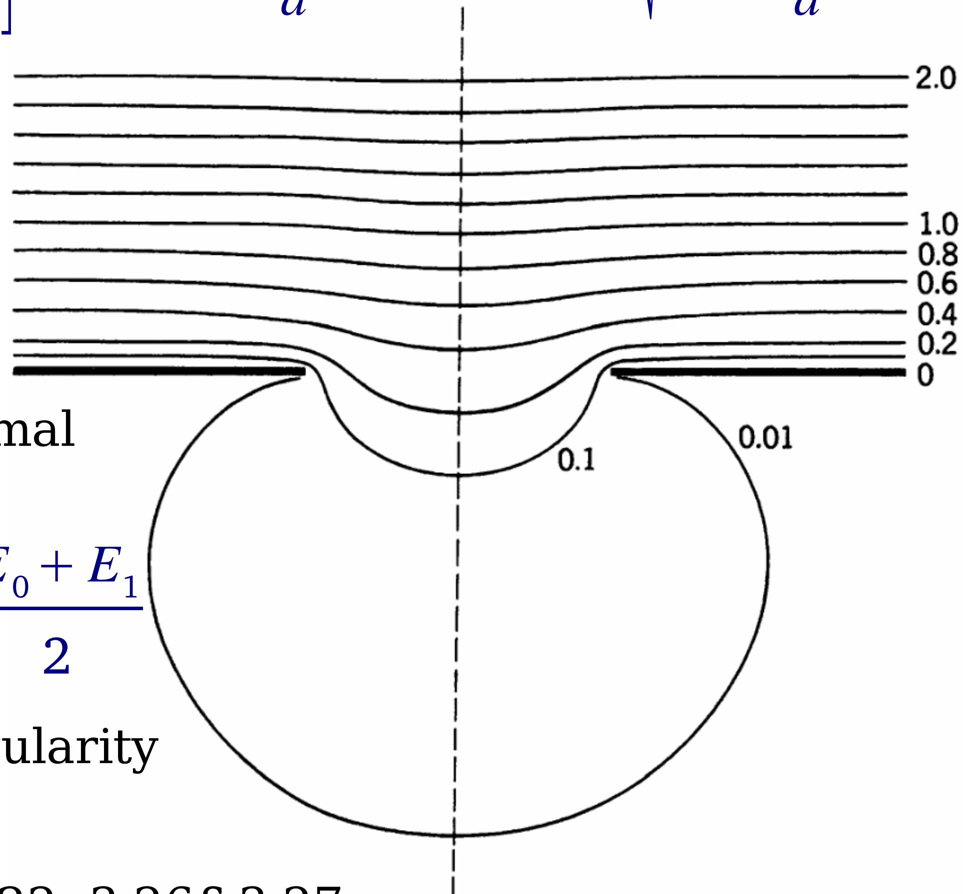
- The idea that a small hole in a plane conducting sheet is equivalent far from the opening to a dipole normal to the surface is important in discussing the consequences of such openings in the walls of waveguides and cavities.

- The added potential in the neighborhood of the opening

$$\Phi^{(1)} = \frac{a^2}{\pi}(E_0 - E_1)\int_0^\infty j_1(ka)\,J_0(k\rho)\,e^{-k|z|}\,dk = \frac{a}{\pi}(E_0 - E_1)\left[\sqrt{\frac{R-\lambda}{2}} - \frac{|z|}{a}\tan^{-1}\sqrt{\frac{2}{R+\lambda}}\right]$$

$$\Rightarrow \quad \Phi^{(1)}(\rho = 0) = \frac{a}{\pi}(E_0 - E_1)\left[1 - \frac{|z|}{a}\tan^{-1}\frac{a}{|z|}\right] \Leftarrow \quad \lambda = \frac{\rho^2 + z^2}{a^2} - 1, \quad R = \sqrt{\lambda^2 + \frac{4z^2}{a^2}}$$

$$\Rightarrow \begin{cases} \Phi^{(1)} \to \dfrac{a^3}{3\pi z^2}(E_0 - E_1) & \text{for } |z| \gg a \\ \Phi^{(1)} \to \dfrac{E_0 - E_1}{\pi}\sqrt{a^2 - \rho^2} & \text{for } |z| \to 0 \end{cases}$$



- The tangential (a radial field) and the normal electric field in the opening

$$\mathbf{E}_{\tan}(\rho, 0) = \frac{E_0 - E_1}{\pi}\frac{\boldsymbol{\rho}}{\sqrt{a^2 - \rho^2}}, \quad E_z(\rho, 0) = -\frac{E_0 + E_1}{2}$$

- the magnitude of **E** has a square root singularity at the edge of the opening.

- Selected problems: 3.3, 3.7, 3.12, 3.19, 3.22, 3.26&3.27