Chapter 2 Boundary-Value Problems in Electrostatics I

- The correct Green function is not necessarily easy to be found.

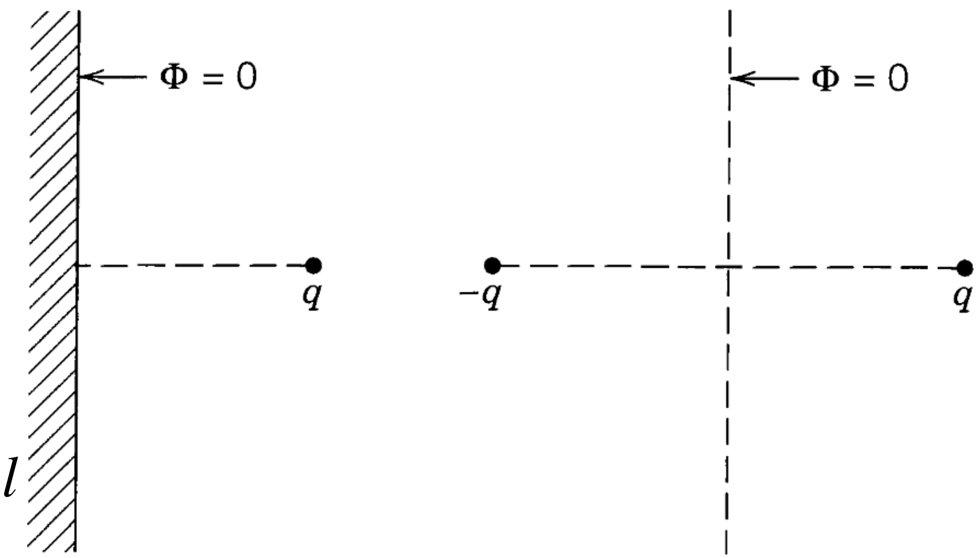
- 3 techniques to electrostatic boundary value problems:

(1) the method of images

(2) expansion in orthogonal functions

(3) finite element analysis (numerical method)

§2.1 Method of Images

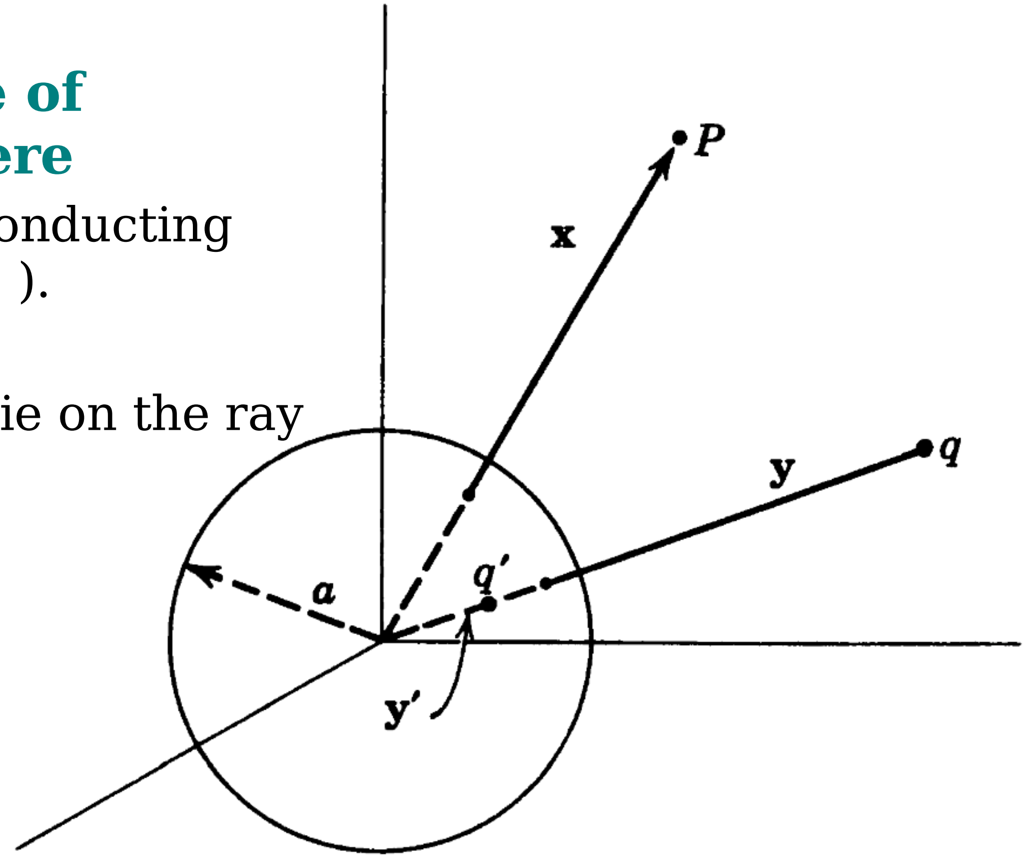


- The method deals with the problem of point charges in the presence of boundary surfaces, eg, conductors either grounded or held at fixed potentials.

- Infer from the geometry of the situation that some suitably placed charges, *external to the region of interest*, can simulate the boundary conditions. The charges are called *image charges*.

- The method replaces the actual problem with boundaries by an enlarged region with image charges but not boundaries.

§2.2 Point Charge in the Presence of a Grounded Conducting Sphere



- A point charge is outside a grounded conducting sphere. Find the potential ($\Phi(|\mathbf{x}|=a)=0$).

- By symmetry the image charge q' will lie on the ray from the origin to the charge q, then

$$\Phi(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\mathbf{x}-\mathbf{y}|}+\frac{q'}{|\mathbf{x}-\mathbf{y}'|}\right)$$

$$=\frac{1}{4\pi\epsilon_0}\left(\frac{q}{|x\,\mathbf{n}-y\,\mathbf{n}'|}+\frac{q'}{|x\,\mathbf{n}-y'\,\mathbf{n}'|}\right)$$

$$\Rightarrow\quad 4\pi\epsilon_0\Phi(x=a)=\frac{q}{a\left|\mathbf{n}-y\,\mathbf{n}'/a\right|}+\frac{q'}{y'\left|\mathbf{n}'-a\,\mathbf{n}/y'\right|}=0$$

$$\Rightarrow\quad \frac{q}{a}=-\frac{q'}{y'},\quad \frac{y}{a}=\frac{a}{y'}\quad\Rightarrow\quad \boxed{y'=\frac{a^2}{y},\quad q'=-\frac{a}{y}q}$$

- As q is closer to the sphere, the q' grows and moves out from the center of the sphere.

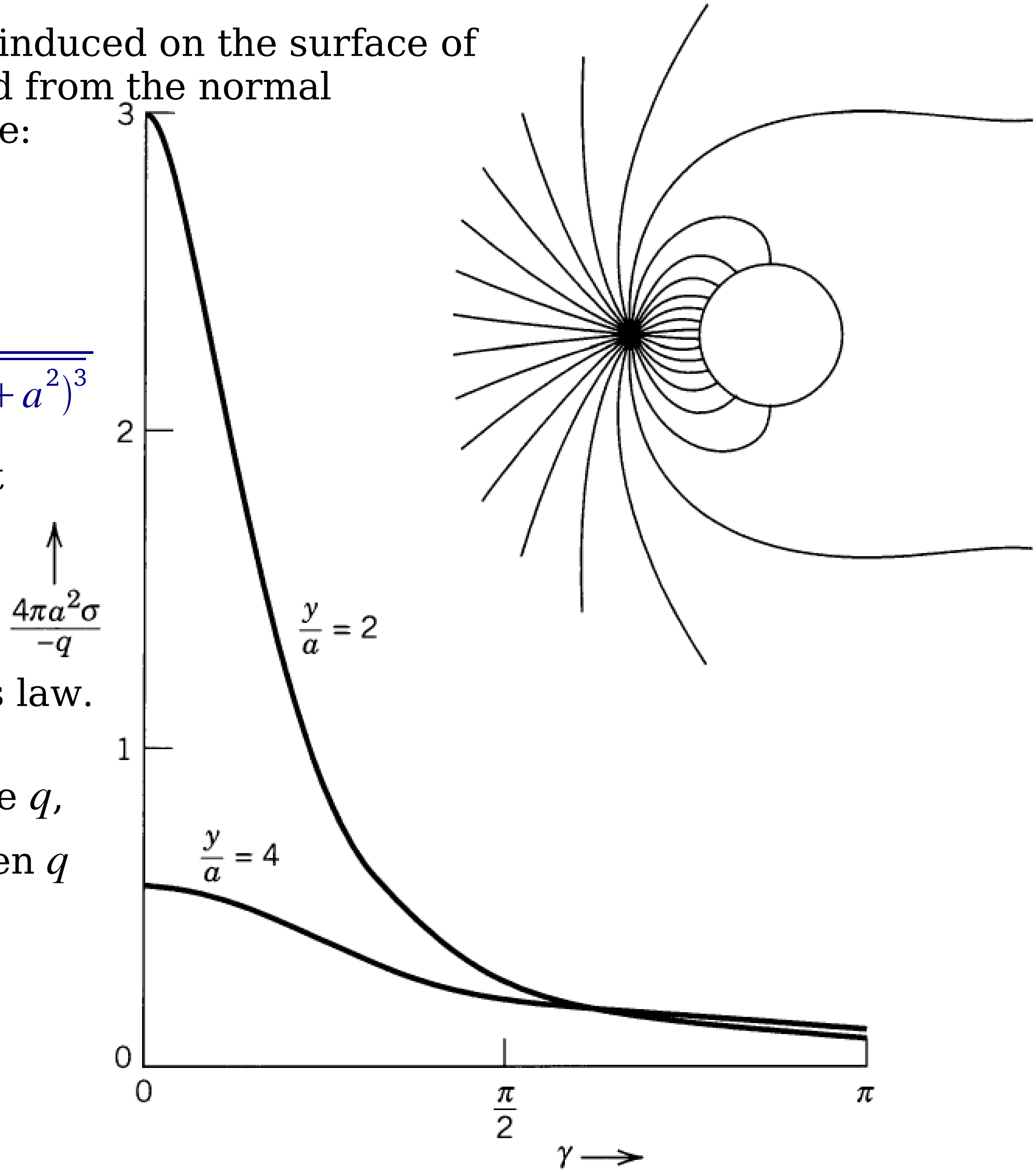
- When q is just outside the surface of the sphere, q' is equal and opposite in and lies just beneath the surface.

- The actual charge density induced on the surface of the sphere can be calculated from the normal derivative of Φ at the surface:

$$\sigma = -\epsilon_0 \left. \frac{\partial \Phi}{\partial x} \right|_{x=a}$$

$$= \frac{q}{4\pi a^2} \frac{a(a^2 - y^2)}{\sqrt{(y^2 - 2ay\cos\gamma + a^2)^3}}$$

- It is easy to show by direct integration that the total induced charge on the sphere is equal to the magnitude of the image charge according to Gauss's law.



- For the force on the charge q, write down the force between q and q'

$$y - y' = y\left(1 - \frac{a^2}{y^2}\right)$$

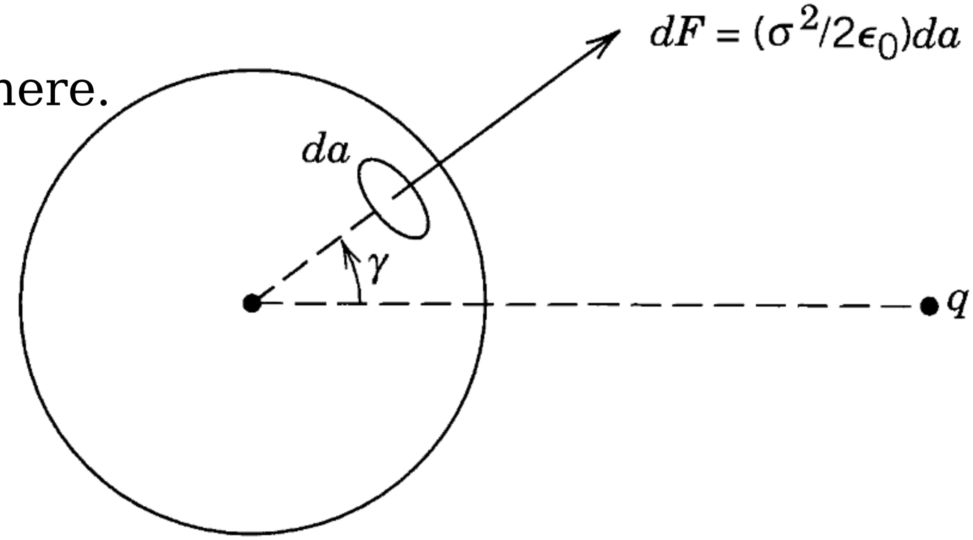
$$\Rightarrow \quad |\mathbf{F}| = \frac{q^2}{4\pi\epsilon_0} \frac{a\,y}{(y^2 - a^2)^2}$$

- The alternative method for the force is to calculate the total force acting on the surface of the sphere

$$|\mathbf{F}| = \int \mathrm{d}F = \int \frac{\sigma^2}{2\epsilon_0}\,\mathrm{d}a = \frac{q^2 (y^2 - a^2)^2}{32\pi^2\epsilon_0} \int \frac{\cos\gamma}{(y^2 + a^2 - 2ay\cos\gamma)^3}\,\mathrm{d}\Omega = \frac{q^2}{4\pi\epsilon_0}\frac{ay}{(y^2 - a^2)^2}$$

- The whole discussion has been based on the understanding that q is outside the sphere.

Actually, the results apply equally for q inside the sphere.



§2.3 Point Charge in the Presence of a Charged, Insulated, Conducting Sphere

- Consider an insulated conducting sphere with total charge Q in the presence of a point charge q.

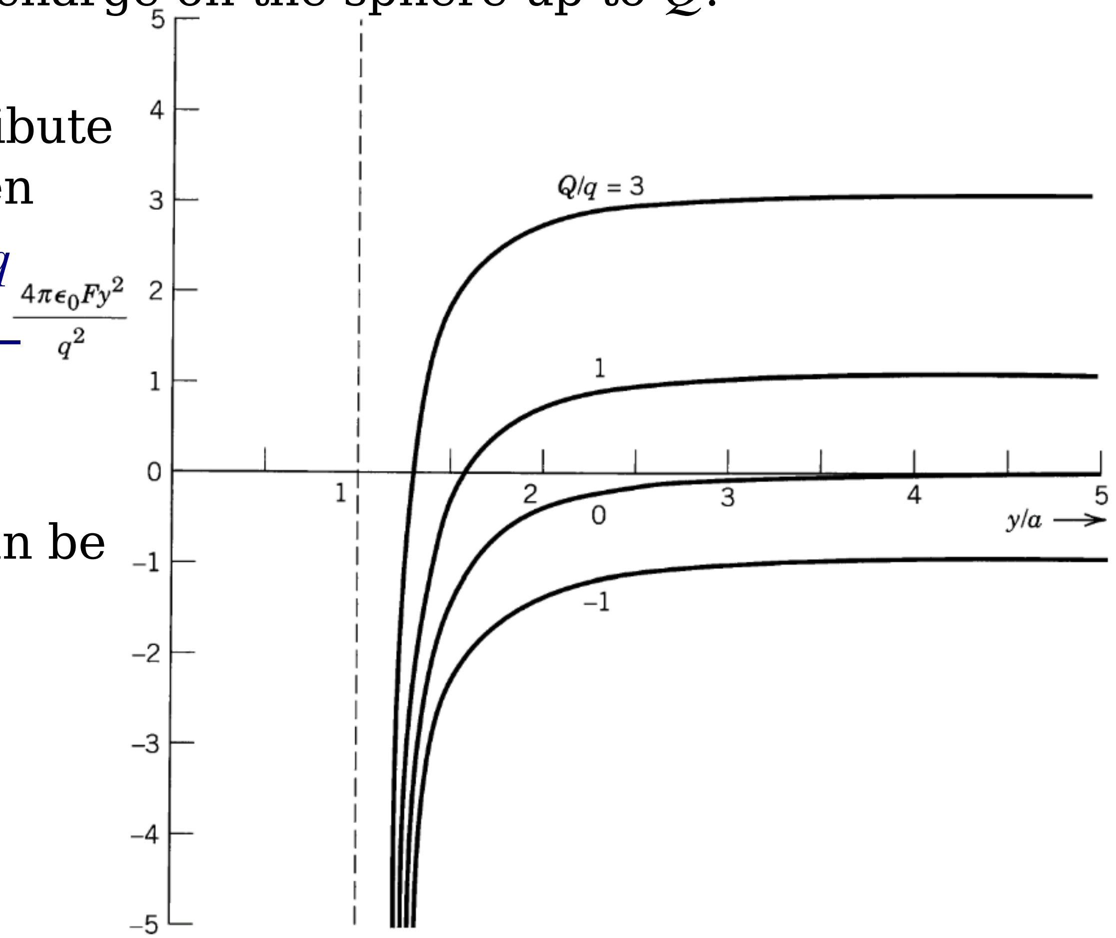
- Start with the grounded conducting sphere (with its charge q' distributed over its surface). Then disconnect the ground wire and add to the sphere an amount of charge (Q - q'). This brings the total charge on the sphere up to Q.

- The added charge (Q - q') will distribute itself uniformly over the surface. Then

$$4\pi\epsilon_0 \Phi(\mathbf{x}) = \frac{q}{|\mathbf{x}-\mathbf{y}|} - \frac{\frac{a}{y}q}{\left|\mathbf{x}-\frac{a^2}{y^2}\mathbf{y}\right|} + \frac{Q+\frac{a}{y}q}{|\mathbf{x}|}$$

- The force acting on the charge q can be written down from Coulomb's law

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q}{y^2}\left[Q - \frac{q a^3(2y^2-a^2)}{y(y^2-a^2)^2}\right]\frac{\mathbf{y}}{y}$$



- In the limit of $y \gg a$, the force reduces to the usual Coulomb's law for two small charged bodies. But close to the sphere the force is modified because of the induced charge distribution on the surface of the sphere.

- If the sphere is charged oppositely to q, or is uncharged, the force is attractive at all distances.

- Even if the charge Q is the same sign as q, the force becomes attractive at very close distances.

- In the limit of $Q \gg q$, the point of zero force (unstable equilibrium point) is very close to the sphere, at

$$y \simeq a\left(1+\frac{1}{2}\sqrt{\frac{q}{Q}}\right)$$

- This example explains why an excess of charge on the surface does not leave the surface because of mutual repulsion of the individual charges.

- As soon as an element of charge is removed from the surface, the image force tends to attract it back.

§2.4 Point Charge Near a Conducting Sphere at Fixed Potential

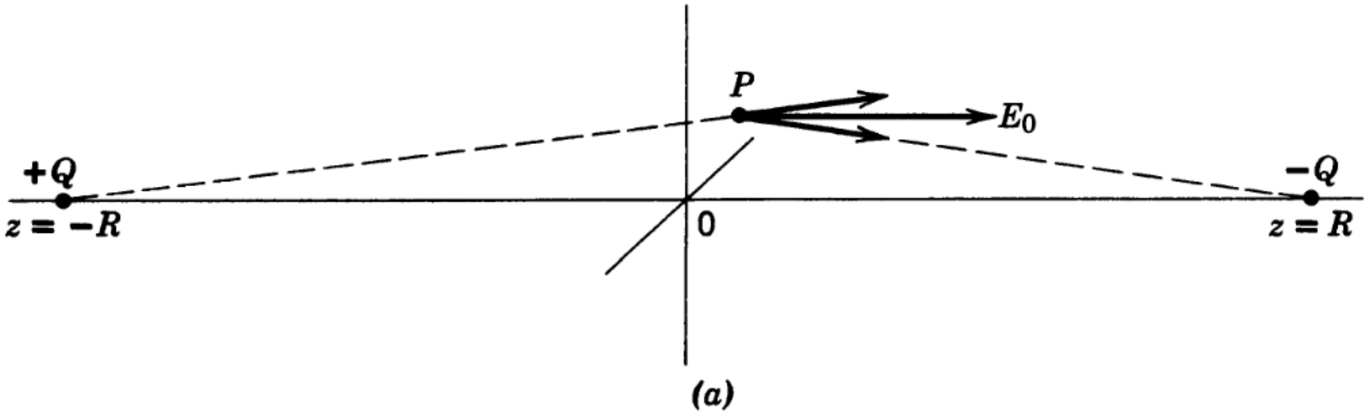
- The potential is the same as for the charged sphere, except that the charge (Q - q') at the center is replaced by a charge $4\pi\epsilon_0 Va$

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\mathbf{x}-\mathbf{y}|} - \frac{a\,q}{y\left|\mathbf{x}-\frac{a^2}{y^2}\mathbf{y}\right|}\right] + \frac{a}{|\mathbf{x}|}V \quad \Rightarrow \quad \mathbf{F} = \frac{q}{y^2}\left[a\,V - \frac{1}{4\pi\epsilon_0}\frac{q\,a\,y^3}{(y^2-a^2)^2}\right]\frac{\mathbf{y}}{y}$$

- For $4\pi\epsilon_0 aV \gg q$, the unstable equilibrium point: $y \simeq a\left(1+\frac{1}{2}\sqrt{\frac{q}{4\pi\epsilon_0 a V}}\right)$

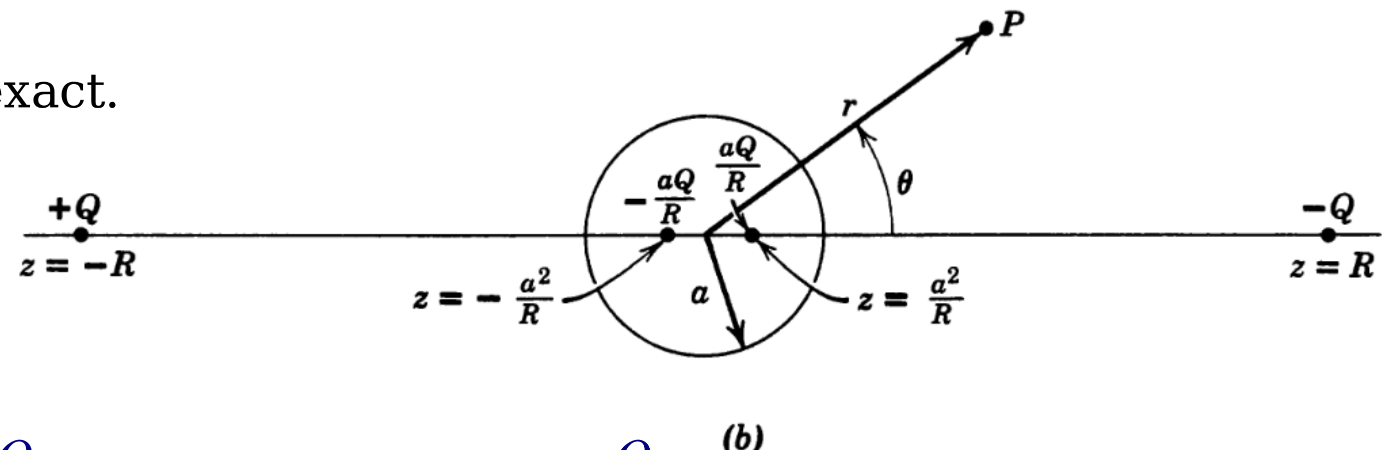
§2.5 Conducting Sphere in a Uniform Electric Field by Method of Images

- Consider a conducting sphere of radius a in a uniform electric field. A uniform field can be thought of as being produced by appropriate positive and negative charges at infinity.



- The electric field near the origin: $E_0 \simeq \frac{1}{2\pi\epsilon_0}\frac{Q}{R^2}$

- In the limit as R, $Q \to \infty$ with Q/R^2 constant, this approximation becomes exact.

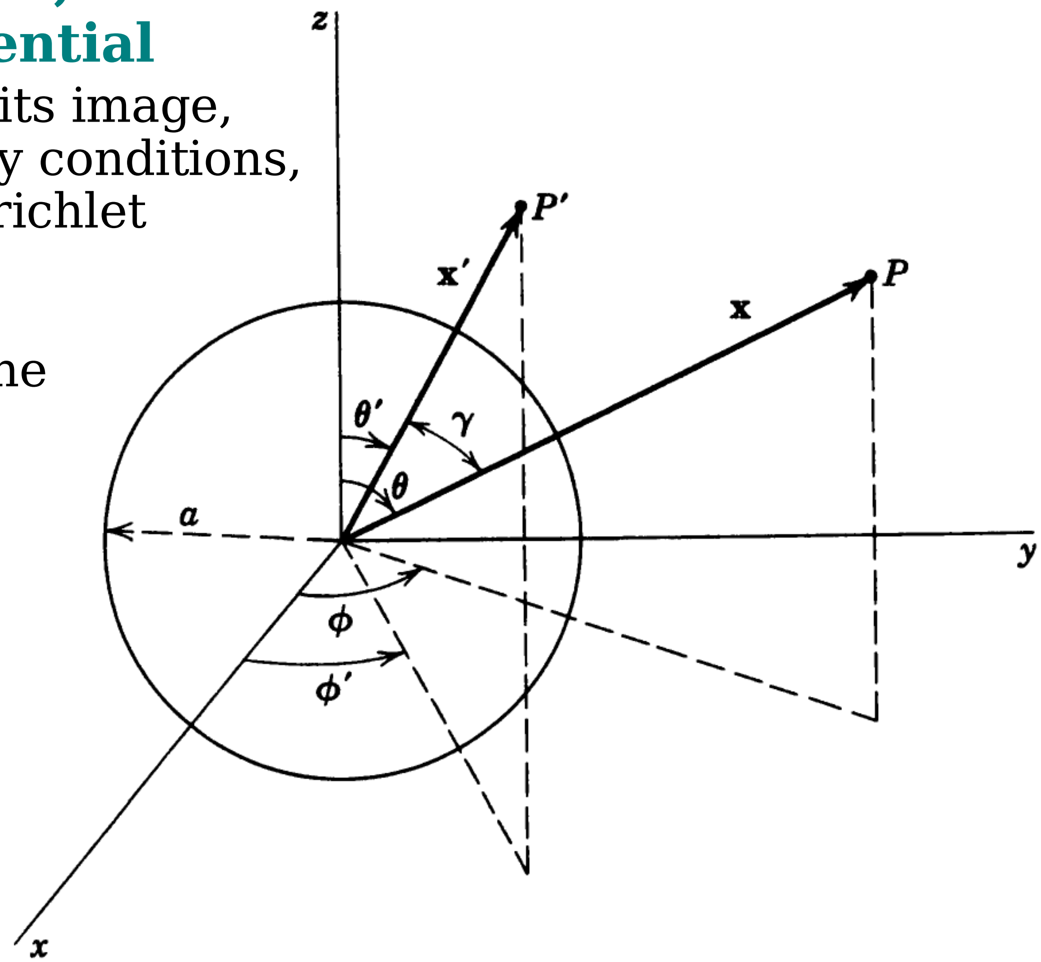


- A conducting sphere is placed at the origin, the potential will be that due to the charges and their images

$$4\pi\epsilon_0\Phi = \frac{Q}{\sqrt{r^2+R^2+2rR\cos\theta}} - \frac{aQ}{\sqrt{r^2R^2+a^4+2a^2rR\cos\theta}}$$
$$-\frac{Q}{\sqrt{r^2+R^2-2rR\cos\theta}} + \frac{aQ}{\sqrt{r^2R^2+a^4-2a^2rR\cos\theta}}$$

- For $R \gg r$: $4\pi\epsilon_0 \Phi = -\frac{2Q}{R^2}\left(r - \frac{a^3}{r^2}\right)\cos\theta + \cdots$
- To the limit $2Q/4\pi\epsilon_0 R^2$ becomes the applied uniform field: $\Phi = -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta$
- The 1st term ($-E_0 z$) is the potential of a uniform field. The 2nd is the potential due to the induced surface-charge density or, equivalently, the image charges.
- The image charges form a dipole of strength $D = \frac{a}{R} Q \times 2\frac{a^2}{R} = 4\pi\epsilon_0 E_0 a^3$
- The induced surface-charge density $\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial r}\Big|_{r=a} = 3\epsilon_0 E_0 \cos\theta$
- the surface integral of this charge density vanishes, so that there is no difference between a grounded and an insulated sphere.

§2.6 Green Function for the Sphere; General Solution for the Potential



- The potential due to a unit source and its image, chosen to satisfy homogeneous boundary conditions, is the Green function appropriate for Dirichlet or Neumann boundary conditions.

- For Dirichlet boundary conditions on the sphere, the Green function for a unit source and its image is

$$G(\mathbf{x},\mathbf{x}')=\frac{1}{|\mathbf{x}-\mathbf{x}'|}-\frac{a\,x'}{|x'^2\mathbf{x}-a^2\mathbf{x}'|}$$

$$=\frac{1}{\sqrt{x^2+x'^2-2\,x\,x'\cos\gamma}}$$

$$-\frac{a}{\sqrt{x^2x'^2+a^4-2\,x\,x'a^2\cos\gamma}}$$

- The symmetry in the variables is obvious, as is the condition that $G = 0$ if either **x** or **x**' is on the surface of the sphere.

- For solutions of a Poisson equation we need not only G, but also $\partial G/\partial n'$

$$\left.\frac{\partial G}{\partial n'}\right|_{x'=a}=\frac{a^2-x^2}{a\sqrt{(x^2+a^2-2\,a\,x\cos\gamma)^3}}$$

- The condition is essentially the induced surface-charge density.

- The solution of the Laplace equation *outside* a sphere with the potential specified on its surface is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi} \int \Phi(a, \theta', \phi') \frac{a(x^2 - a^2)}{\sqrt{(x^2 + a^2 - 2ax\cos\gamma)^3}} \, \mathrm{d}\Omega'$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$

- For the *interior* problem, $\left.\frac{\partial G}{\partial n'}\right|_{x'=a} = \frac{x^2 - a^2}{a\sqrt{(x^2 + a^2 - 2ax\cos\gamma)^3}}$

$$\Rightarrow \quad \Phi(\mathbf{x}) = \frac{1}{4\pi} \int \Phi(a, \theta', \phi') \frac{a(a^2 - x^2)}{\sqrt{(x^2 + a^2 - 2ax\cos\gamma)^3}} \, \mathrm{d}\Omega'$$

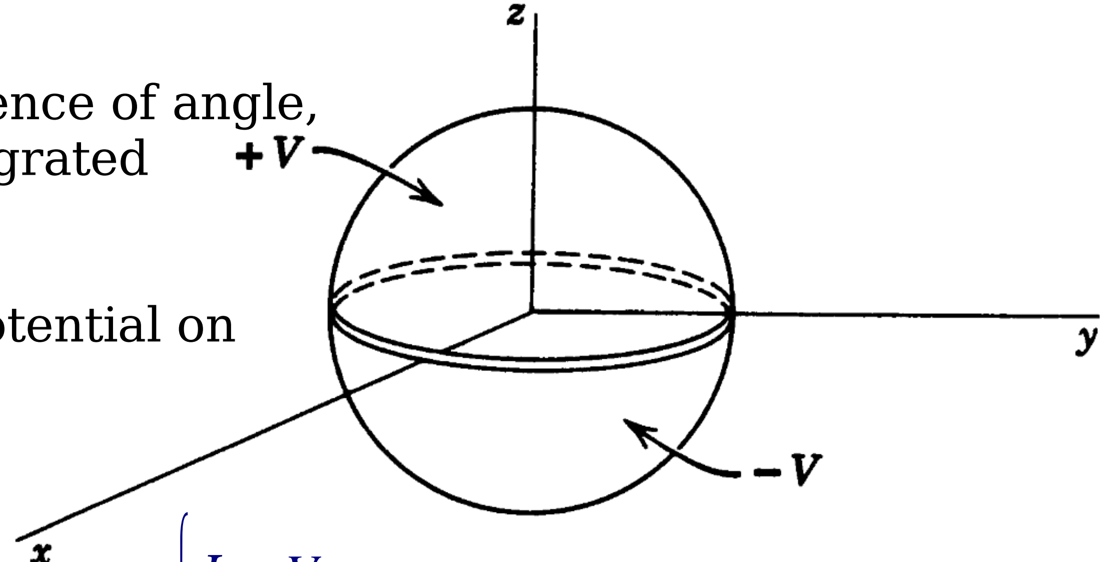
- For a problem with a charge distribution, we must add to the potential integral the appropriate charge density with the Green function.

§2.7 Conducting Sphere with Hemispheres at Different Potentials

- The solution for the potential is

$$\Phi(x,\theta,\phi)=\frac{V}{4\pi}\int_0^{2\pi}\mathrm{d}\phi'\left[\int_0^1\frac{a(x^2-a^2)\,\mathrm{d}\cos\theta'}{\sqrt{(x^2+a^2-2ax\cos\gamma)^3}}-\int_{-1}^0\frac{a(x^2-a^2)\,\mathrm{d}\cos\theta'}{\sqrt{(x^2+a^2-2ax\cos\gamma)^3}}\right]$$

$$=\frac{Va(x^2-a^2)}{4\pi}\int_0^{2\pi}\mathrm{d}\phi'\int_0^1\mathrm{d}\cos\theta'\left[(x^2+a^2-2ax\cos\gamma)^{-3/2}-(x^2+a^2+2ax\cos\gamma)^{-3/2}\right]$$



- Because of the complicated dependence of angle, the integral cannot in general be integrated in closed form.

- As a special case we consider the potential on the positive z axis

$$\cos\gamma=\cos\theta' \quad \Leftarrow \quad \theta=0$$

$$\Rightarrow \quad \Phi(z)=V\left(1-\frac{z^2-a^2}{z\sqrt{z^2+a^2}}\right) \quad \Rightarrow \quad \begin{cases}\Phi=V & z=a\\ \Phi\simeq 3Va^2/2z^2 & z\to\infty\end{cases}$$

- In the absence of a closed expression for the integral, we can expand the denominator in power series and integrate term by term

$$\alpha=\frac{ax}{x^2+a^2}$$

$$\Phi=\frac{Va(x^2-a^2)}{4\pi\sqrt{(x^2+a^2)^3}}\int_0^{2\pi}\mathrm{d}\phi'\int_0^1\mathrm{d}\cos\theta'\left[(1-2\alpha\cos\gamma)^{-3/2}-(1+2\alpha\cos\gamma)^{-3/2}\right]$$

- in the expansion of the radicals only odd powers of $\alpha\cos\gamma$ will appear

$$(1-2\,\alpha\cos\gamma)^{-3/2}-(1+2\,\alpha\cos\gamma)^{-3/2}=6\,\alpha\cos\gamma+35\,\alpha^3\cos^3\gamma+\cdots$$

$$\Rightarrow \quad \left\{\begin{array}{l} \int_0^{2\pi} \mathrm{d}\,\phi'\int_0^1 \cos\gamma\;\; \mathrm{d}\cos\theta' = \pi\cos\theta \\ \int_0^{2\pi} \mathrm{d}\,\phi'\int_0^1 \cos^3\gamma\; \mathrm{d}\cos\theta' = \dfrac{\pi}{4}\cos\theta\,(3-\cos^2\theta) \end{array}\right.$$

$$\Rightarrow \quad \Phi(x,\theta,\phi)=\frac{3\,a\,(x^2-a^2)\,V}{2\sqrt{(x^2+a^2)^3}}\,\alpha\cos\theta\left[1+\frac{35}{24}\,\alpha^2(3-\cos^2\theta)+\cdots\right] \quad \Leftarrow \quad \text{in} \quad \alpha$$

$$=\frac{3}{2}\,\frac{a^2}{x^2}\,V\left[\cos\theta-\frac{7}{12}\,\frac{a^2}{x^2}\left(\frac{5}{2}\cos^3\theta-\frac{3}{2}\cos\theta\right)+\cdots\right] \quad \Leftarrow \quad \text{in} \quad \frac{a^2}{x^2}$$

- only odd powers of $\cos\theta$ appear, as required by the symmetry of the problem.

- For large values of x/a this expansion converges rapidly and so is a useful representation for the potential.

- It is easily verified that, for $\cos\theta=1$, this expression agrees with the expansion of the expression for the potential on the axis.

§2.8 Orthogonal Functions and Expansions

- The orthogonal set chosen depends on the symmetries or near symmetries involved.

- Consider an interval (a, b) in a variable ξ with a set of real or complex *orthonormal* functions $U_n(\xi)$. The orthogonality condition is

$$\int_a^b U_n^*(\xi)\, U_m(\xi)\, \mathrm{d}\xi = \delta_{mn}$$

- An arbitrary function can be expanded in a series of the orthonormal functions

$$f(\xi) = \sum_{n=1}^{\infty} a_n U_n(\xi) \quad \Leftarrow \quad \textit{completeness} \text{ of the function set}$$

$$= \sum_{n=1}^{\infty} \left[\int_a^b U_n^*(\xi')\, f(\xi')\, \mathrm{d}\xi' \right] U_n(\xi) \quad \Leftarrow \quad a_n = \int_a^b U_n^*(\xi)\, f(\xi)\, \mathrm{d}\xi$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} U_n^*(\xi')\, U_n(\xi) = \delta(\xi' - \xi) \quad \Leftarrow \quad \textit{closure relation} \text{ or } \textit{completeness}$$

- The most famous orthogonal functions are the sines and cosines, an expansion in terms of them being a *Fourier series*.

- The orthonormal functions are $\sqrt{\dfrac{2}{a}} \sin \dfrac{2\pi m x}{a}, \quad \sqrt{\dfrac{2}{a}} \cos \dfrac{2\pi m x}{a} \quad \text{for} \quad x \in [-\dfrac{a}{2}, \dfrac{a}{2}]$

- the constant function is $\dfrac{1}{\sqrt{a}} \quad \text{for} \quad m = 0$

- A function is customarily written in the form:

$$f(x) = \frac{1}{2} A_0 + \sum_{m=1}^{\infty} \left[A_m \cos \frac{2\pi m x}{a} + B_m \sin \frac{2\pi m x}{a} \right]$$

where $\quad A_m = \frac{2}{a} \int_{-a/2}^{a/2} f(x) \cos \frac{2\pi m x}{a} \,\mathrm{d}x, \quad B_m = \frac{2}{a} \int_{-a/2}^{a/2} f(x) \sin \frac{2\pi m x}{a} \,\mathrm{d}x$

- Suppose that the space is 2d, and the variable ξ ranges over the interval (a, b) while the variable η has the interval (c,d). The orthonormal functions in each dimension are $U_n(\xi)$ and $V_m(\eta)$. Then the expansion of an arbitrary function is

$$f(\xi, \eta) = \sum_n \sum_m a_{nm} U_n(\xi) V_m(\eta) \quad \Leftarrow \quad a_{nm} = \int_a^b \mathrm{d}\xi \int_c^d \mathrm{d}\eta\, U_n^*(\xi) V_m^*(\eta) f(\xi, \eta)$$

- $(a, b) \;\to\; (-\infty, \infty) \quad \Rightarrow \quad U_m(x) \;\to\; U(m, x)$

$$\Rightarrow \quad \int_a^b U_n^*(x) U_m(x) \,\mathrm{d}x \;\to\; \int_{-\infty}^{\infty} U^*(n, x) U(m, x) \,\mathrm{d}x = \delta(m-n) \;\leftarrow\; \delta_{mn}$$

- For *Fourier integral*, start with: $U_m(x) = \frac{1}{\sqrt{a}} e^{i \frac{2 m \pi x}{a}} \quad \Leftarrow \quad \begin{cases} m = 0, \pm 1, \pm 2, \cdots \\ x \in (-a/2, a/2) \end{cases}$

$$\Rightarrow \quad f(x) = \frac{1}{\sqrt{a}} \sum_{m=-\infty}^{\infty} A_m e^{i \frac{2 m \pi x}{a}} \quad \Leftarrow \quad A_m = \frac{1}{\sqrt{a}} \int_{-a/2}^{a/2} e^{-i \frac{2 m \pi x'}{a}} f(x') \,\mathrm{d}x'$$

- $a \to \infty \quad \Rightarrow \quad \begin{cases} \dfrac{2\pi m}{a} \to k \\ \sum\limits_m \to \int_{-\infty}^{\infty} \mathrm{d}\, m \to \dfrac{a}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\, k \\ A_m \to \sqrt{\dfrac{2\pi}{a}} A(k) \end{cases}$

$$\Rightarrow \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{i k x}\, \mathrm{d}\, k \quad \Leftarrow \quad A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i k x} f(x)\, \mathrm{d}\, x$$

$$\Rightarrow \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k-k')x}\, \mathrm{d}\, x = \delta(k-k'), \qquad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i k (x-x')}\, \mathrm{d}\, k = \delta(x-x')$$

(orthogonality condition) (completeness relation)

- The 2 continuous variables x and k are complete equivalent.

§2.9 Separation of Variables; Laplace Equation in Rectangular Coordinates

- Equations involving the three-dimensional Laplacian operator are known to be separable in 11 different coordinate systems.

- discuss only 3 of these — rectangular, spherical, and cylindrical.

- The Laplace equation in rectangular coordinates: $\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2}+\frac{\partial^2 \Phi}{\partial z^2}=0$

- assume $\Phi(x, y, z)=X(x)\, Y(y)\, Z(z) \quad \Rightarrow \quad \frac{1}{X}\frac{d^2 X}{d x^2}+\frac{1}{Y}\frac{d^2 Y}{d y^2}+\frac{1}{Z}\frac{d^2 Z}{d z^2}=0$

- hold for arbitrary values of the independent coordinates, each of the 3 terms must be separately constant:

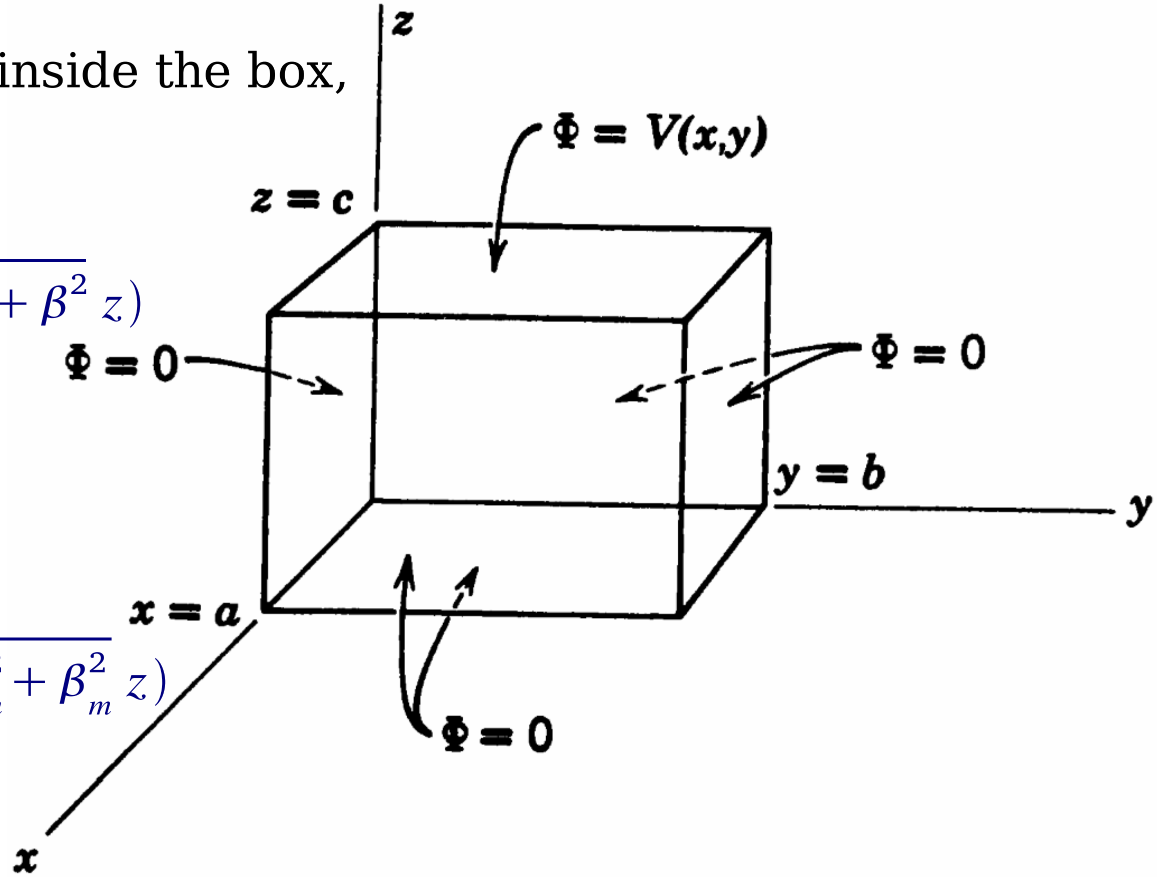
$$\frac{1}{X}\frac{d^2 X}{d x^2}=-\alpha^2, \quad \frac{1}{Y}\frac{d^2 Y}{d y^2}=-\beta^2, \quad \frac{1}{Z}\frac{d^2 Z}{d z^2}=\alpha^2+\beta^2 \quad \Rightarrow \quad \Phi=e^{\pm i\alpha x}\, e^{\pm i\beta y}\, e^{\pm\sqrt{\alpha^2+\beta^2}\, z}$$

- By linear superposition, the solution can construct a very large class of solutions to the Laplace equation.

- To determine α and β it is necessary to impose specific boundary conditions on the potential.

- To find the potential everywhere inside the box,

$$\begin{aligned}\Phi(x=0, y, z) &= 0 \\ \Phi(x, y=0, z) &= 0 \\ \Phi(x, y, z=0) &= 0\end{aligned} \quad \Rightarrow \quad \begin{aligned} X &= \sin \alpha x \\ Y &= \sin \beta y \\ Z &= \sinh\left(\sqrt{\alpha^2+\beta^2}\, z\right)\end{aligned}$$



$$\begin{aligned}\Phi(x=a, y, z) &= 0 \\ \Phi(x, y=b, z) &= 0\end{aligned} \quad \Rightarrow \quad \begin{aligned}\alpha_n &= \frac{n\pi}{a} \\ \beta_m &= \frac{m\pi}{b}\end{aligned}$$

$$\Rightarrow \quad \Phi_{nm} = \sin(\alpha_n x)\sin(\beta_m y)\sinh\left(\sqrt{\alpha_n^2+\beta_m^2}\, z\right)$$

$$\Rightarrow \quad \Phi(x, y, z) = \sum_{n,m=1}^{\infty} A_{nm}\Phi_{nm}$$

$$= \sum_{n,m=1}^{\infty} A_{nm}\sin(\alpha_n x)\sin(\beta_m y)\sinh\left(\sqrt{\alpha_n^2+\beta_m^2}\, z\right)$$

$$\Phi(x, y, z=c) = V(x, y) \quad \Rightarrow \quad V(x, y) = \sum_{n,m=1}^{\infty} A_{nm}\sin(\alpha_n x)\sin(\beta_m y)\sinh\left(\sqrt{\alpha_n^2+\beta_m^2}\, c\right)$$

$$\Rightarrow \quad A_{nm} = \frac{4}{a\, b \sinh\left(c\sqrt{\alpha_n^2+\beta_m^2}\right)} \int_0^a dx \int_0^b dy\, V(x, y)\sin(\alpha_n x)\sin(\beta_m y)$$

- If the rectangular box has potentials different from zero on all 6 sides, the required solution for the potential inside the box can be obtained by a linear superposition of six solutions, one for each side.

- The problem of the solution of the Poisson equation, i.e., the potential inside the box with a charge distribution inside, as well as prescribed boundary conditions on the surface, requires the construction of the appropriate Green function.

§2.10 A 2d Potential Problem; Summation of a Fourier Series

- consider the solution by separation of variables of the 2d Laplace equation in Cartesian coordinates, in which the potential is independent of *z*,

$$\Phi \sim e^{\pm i\alpha x} e^{\pm \alpha y} \quad \Leftarrow \quad \alpha \in \mathbb{R} \text{ or } \mathbb{C}$$

- Boundary conditions:

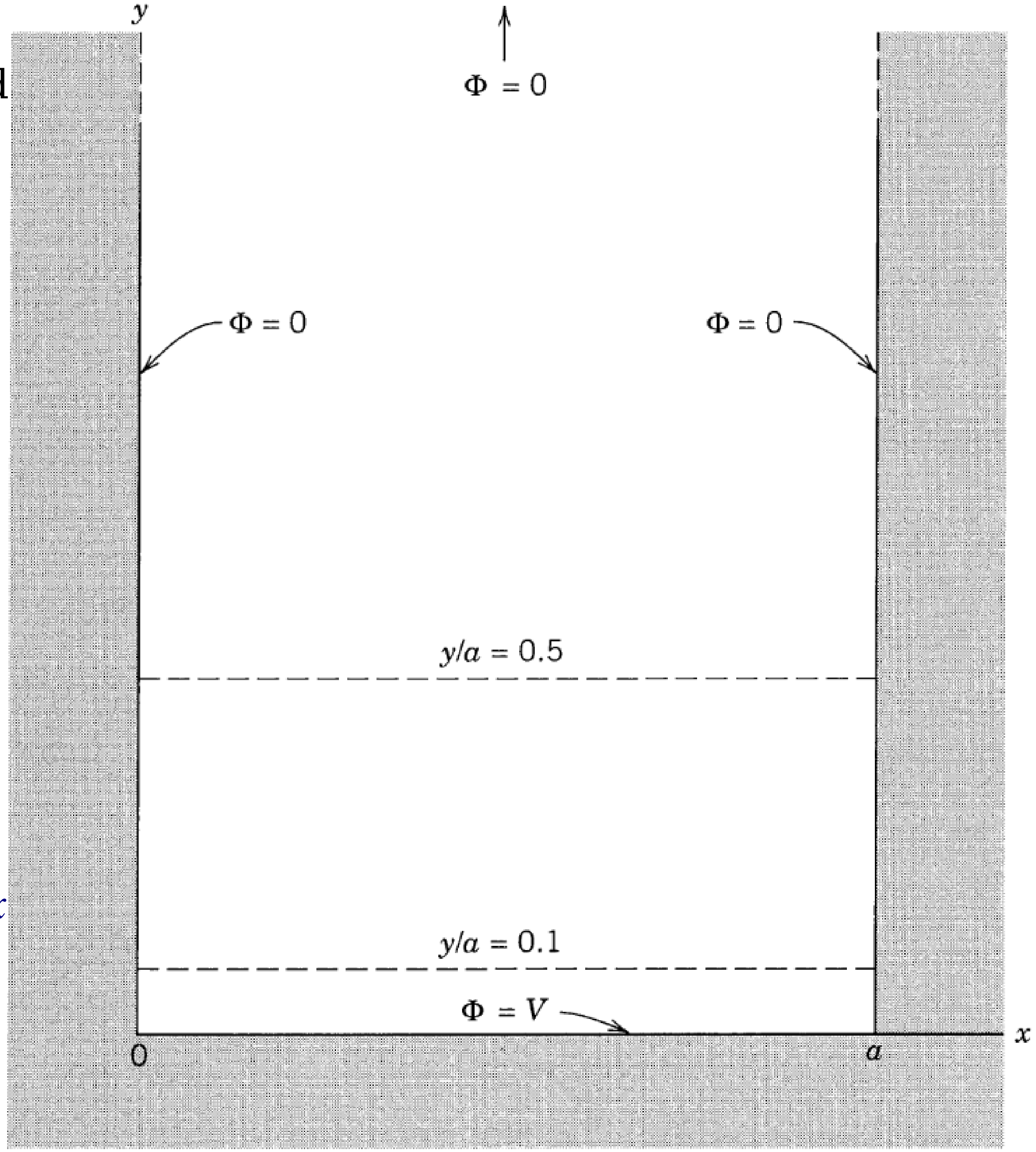
$$\Phi(x=0, y) = \Phi(x=a, y) = 0$$
$$\Phi(x, y=\infty) = 0$$
$$\Phi(x, y=0) = V$$

$$\Rightarrow \quad \Phi \sim e^{-\alpha y} \sin \alpha x \quad \Leftarrow \quad \alpha = \frac{n\pi}{a}$$

$$\Rightarrow \quad \Phi = \sum_{n=1}^{\infty} A_n e^{-\frac{n\pi y}{a}} \sin \frac{n\pi x}{a}$$

$$\Leftarrow \quad A_n = \frac{2}{a} \int_0^a \Phi(x, 0) \sin \frac{n\pi x}{a} \, \mathrm{d}x$$

$$= \frac{4V}{n\pi} \begin{cases} 1 & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$



$$\Rightarrow \quad \Phi(x,y) = \sum_{n\ \mathrm{odd}} \frac{4V}{n\pi} e^{-\frac{n\pi y}{a}} \sin\frac{n\pi x}{a}$$

$$\Rightarrow \quad \Phi \to \frac{4V}{\pi} e^{-\frac{\pi y}{a}} \sin\frac{\pi x}{a} \quad \text{for} \quad y \geqslant \frac{a}{\pi}$$

- the smooth behavior in x of the asymptotic solution sets in for $y \geq a$, regardless of the complexities of $\Phi(x,0)$

- $\sin\theta = \Im\left(e^{i\theta}\right)$

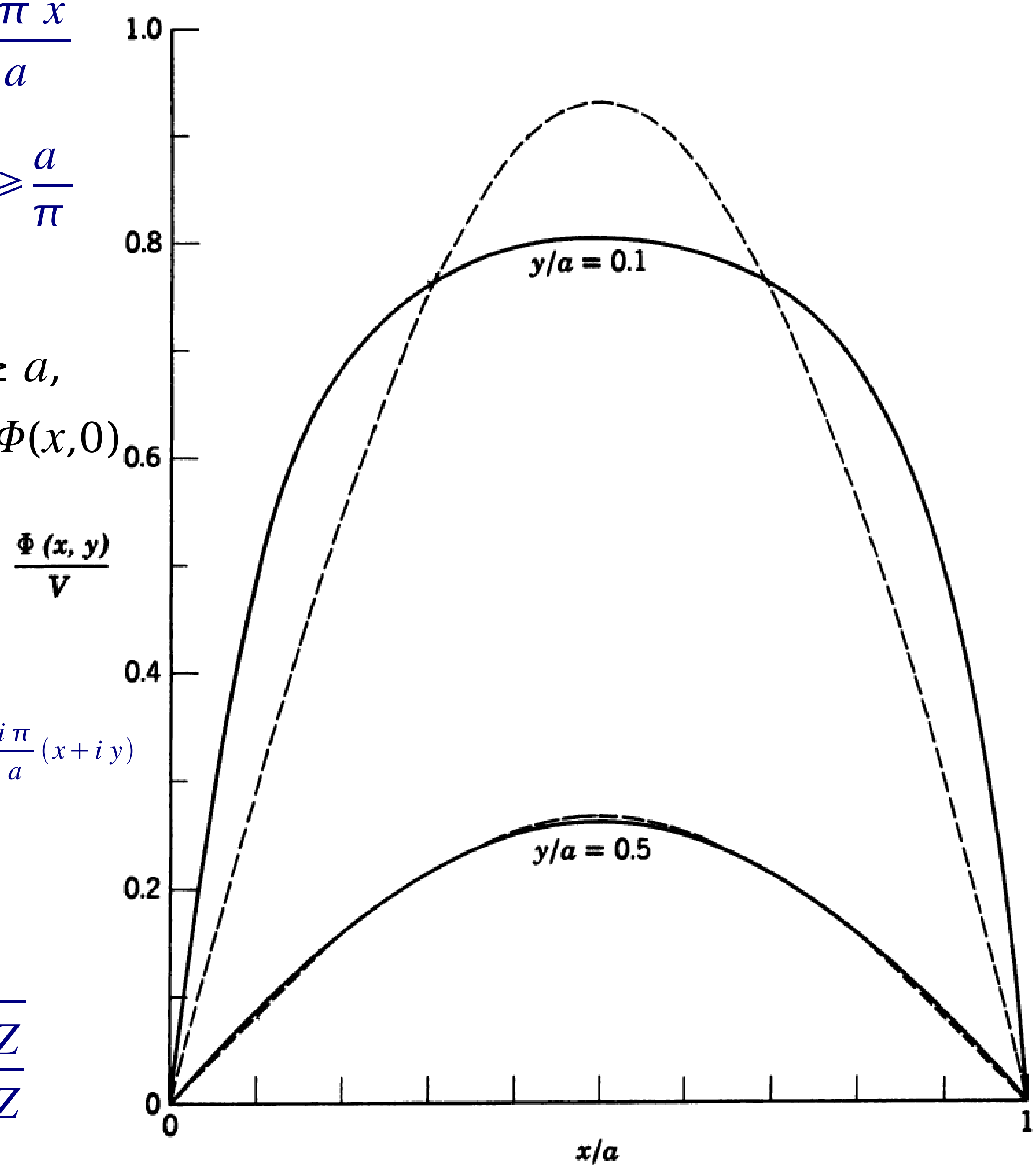
$$\Rightarrow \quad \Phi = \Im\left(\sum_{n\ \mathrm{odd}} \frac{4V}{n\pi} e^{\frac{in\pi}{a}(x+iy)}\right)$$

$$= \frac{4V}{\pi} \Im\left(\sum_{n\ \mathrm{odd}} \frac{Z^n}{n}\right) \quad \Leftarrow \quad Z \equiv e^{\frac{i\pi}{a}(x+iy)}$$

$$\ln(1+Z) = Z - \frac{Z^2}{2} + \frac{Z^3}{3} - \frac{Z^4}{4} + \cdots$$

$$\Rightarrow \quad \sum_{n\ \mathrm{odd}} \frac{Z^n}{n} = \frac{1}{2}\ln\frac{1+Z}{1-Z} = \ln\sqrt{\frac{1+Z}{1-Z}}$$

$$\Rightarrow \quad \Phi(x,y) = \frac{2V}{\pi} \Im\left[\ln\frac{1+Z}{1-Z}\right]$$



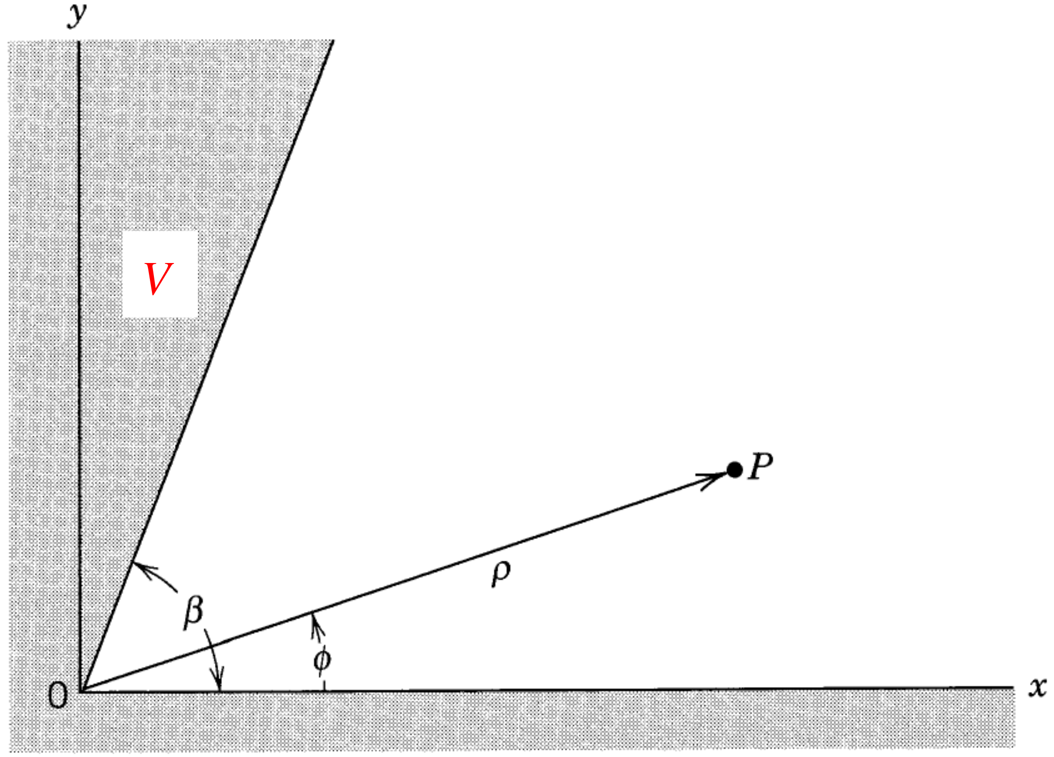
- Since the imaginary part of a logarithm is equal to the phase of its argument,

$$\frac{1+Z}{1-Z}=\frac{1-|Z|^2+2\,i\,\Im(Z)}{|1-Z|^2} \quad \Rightarrow \quad \text{the phase of the argument of } \ln\frac{1+Z}{1-Z}=\tan^{-1}\frac{2\,\Im(Z)}{1-|Z|^2}$$

$$\Rightarrow \quad \Phi(x\,,\,y)=\frac{2\,V}{\pi}\tan^{-1}\left(\frac{\sin\dfrac{\pi\,x}{a}}{\sinh\dfrac{\pi\,y}{a}}\right) \quad \Leftarrow \quad 0\le\tan^{-1}\left(\frac{\sin\dfrac{\pi\,x}{a}}{\sinh\dfrac{\pi\,y}{a}}\right)\le\frac{\pi}{2}$$

- The real or the imaginary part of an analytic function satisfies the Laplace equation in two dimensions as a result of the Cauchy-Riemann equations.

§2.11 Fields and Charge Densities in 2d Corners and Along Edges



- The geometry suggests use of polar rather than Cartesian coordinates.

- The Laplace equation:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}=0$$

- Separation of variables:

$$\Phi(\rho,\phi)=R(\rho)\Psi(\phi)$$

$$\Rightarrow\quad \frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\frac{1}{\Psi}\frac{d^2\Psi}{d\phi}=0$$

$$\Rightarrow\quad \frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)=\nu^2,\quad \frac{1}{\Psi}\frac{d^2\Psi}{d\phi}=-\nu^2$$

$$\Rightarrow\quad \begin{cases} R(\rho)=a\,\rho^{\nu}+b\,\rho^{-\nu} \\ \Psi(\phi)=A\cos(\nu\phi)+B\sin(\nu\phi)\end{cases}\quad \text{but for}\quad \nu=0\ \Rightarrow\ \begin{cases} R(\rho)=a_0+b_0\ln\rho \\ \Psi(\phi)=A_0+B_0\phi\end{cases}$$

- If there is no restriction on ϕ, it is necessary that ν be a positive or negative integer or zero to ensure that the potential is single-valued $\Rightarrow\quad B_0=0$

- The general solution:

$$\Phi(\rho,\phi)=a_0+b_0\ln\rho+\sum_{n=1}^{\infty}\left[a_n\rho^n\sin(n\phi+\alpha_n)+b_n\rho^{-n}\sin(n\phi+\beta_n)\right]$$

- If there is no charge at the origin, all the b_n are zero. If the origin is excluded, the b_n can be different from zero.

- The logarithmic term is equivalent to a line charge on the axis with charge density per unit length $\lambda = -2\pi\epsilon_0 b_0$.

- $\Phi(\rho, 0) = \Phi(\rho, \beta) = V \Rightarrow \quad b_0 = B_0 = b = A = 0, \quad \sin(\nu\beta) = 0 \quad \Leftarrow \quad \nu = \dfrac{m\pi}{\beta}, \quad m = 1, 2, \cdots$

$$\Rightarrow \quad \Phi(\rho, \phi) = V + \sum_{m=1}^{\infty} a_m \rho^{\frac{m\pi}{\beta}} \sin\frac{m\pi\phi}{\beta}$$

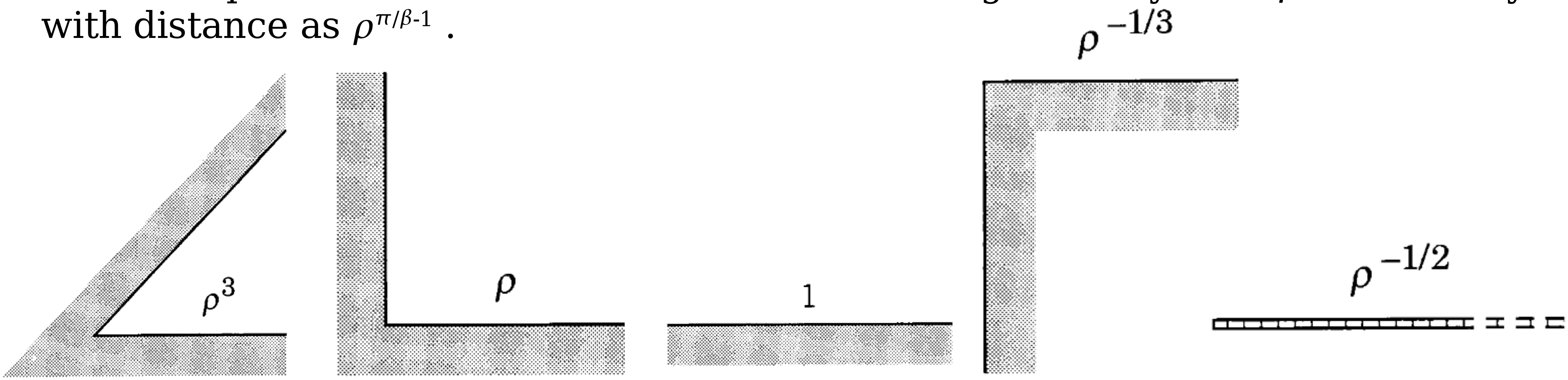
- The undetermined coefficients a_m depend on the potential remote from $\rho = 0$.

- For small enough ρ only the 1st term in the series will be important.

$$\Phi(\rho, \phi) \simeq V + a_1 \rho^{\frac{\pi}{\beta}} \sin\frac{\pi\phi}{\beta} \quad \Rightarrow \quad \left[\begin{aligned} E_\rho &= -\frac{\partial\Phi}{\partial\rho} \simeq -\frac{a_1\pi}{\beta}\rho^{\frac{\pi}{\beta}-1}\sin\frac{\pi\phi}{\beta} \\ E_\phi &= -\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi} \simeq -\frac{a_1\pi}{\beta}\rho^{\frac{\pi}{\beta}-1}\cos\frac{\pi\phi}{\beta} \end{aligned} \right.$$

$$\Rightarrow \quad \sigma(\rho, 0) = \sigma(\rho, \beta) = \epsilon_0 E_\phi(\rho, 0) \simeq -\frac{\epsilon_0 a_1 \pi}{\beta}\rho^{\frac{\pi}{\beta}-1} \quad \Leftarrow \quad \text{surface charge density}$$

- The components of the field and the surface-charge density near $\rho = 0$ all vary with distance as $\rho^{\pi/\beta\text{-}1}$.



- For a very deep corner (small β) the power of ρ becomes very large, and no charge accumulates in such a corner.

- When $\beta > \pi$, the 2d corner becomes an edge and the field and the surface-charge density become singular as $\rho \to 0$.

- The 2d electrostatic considerations apply to many 3d situations, even with time-varying fields.

- The singular behavior of the fields near sharp edges is the reason for the effectiveness of lightning rods.

§2.12 Introduction to Finite Element Analysis for Electrostatics

- FEA encompasses a variety of numerical approaches for the solution of boundary-value problems in physics and engineering.

- Use Galerkin's method for 2d electrostatics as an illustration.

- Consider the Poisson equation, $\nabla^2 \psi = -g$ in a 2d region R, with Dirichlet boundary conditions on the boundary curve C

- The vanishing integral $\int_R [\nabla^2 \psi + g]\, \phi \, \mathrm{d}x \, \mathrm{d}y = 0 \quad \Leftarrow \quad \begin{array}{l} \phi(x, y):\ \text{test function} \\ \phi(x_C, y_C) = 0 \end{array}$

$$\Rightarrow \quad \int_R [\nabla \psi \cdot \nabla \phi - g\, \phi]\, \mathrm{d}x \, \mathrm{d}y = 0 \quad \Leftarrow \quad \text{Green's 1st identity on the 1st term}$$

- Approximate ψ by a finite expansion of a set of localized, linearly independent functions, $\phi_{ij}(x,y)$, with support only in a finite neighborhood of $x=x_i$, $y=y_j$.

- Imagine the region R spanned by a square lattice with spacing h, then a possible choice for ϕ_{ij} is,

$$\phi_{ij}(x, y) = \begin{cases} (1 - |x - x_i|/h)(1 - |y - y_j|/h) & \text{for } |x - x_i| \le h,\ |y - y_j| \le h \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad \sum_{i,j} \phi_{ij}(x, y) = 1 \quad \Rightarrow \quad \psi(x, y) \approx \sum_{i,j}^{N_0} \Psi^{ij} \phi_{ij}(x, y) \quad \Leftarrow \quad N_0 = \begin{array}{l} \text{number of lattice sites} \\ \text{including the boundary} \end{array}$$

- the constant coefficients Ψ^{ij} may be thought of as the approximate values of $\psi(x_i,y_j)$. If the lattice spacing h is small enough, the expansion will be a good approximate to the true ψ, provided the coefficients are chosen properly.

- choose the test function ϕ to be the $(i,j)^{\text{th}}$ function on the expansion set, for $0 \le i, j \le N$ (number of internal sites). Assume $g(x,y)$ varies slowly, then

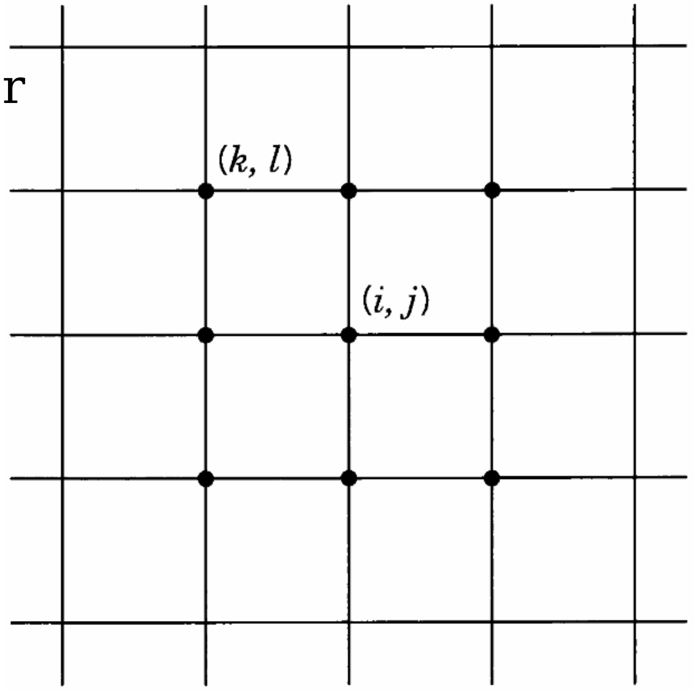
$$\sum_{k,\ell}^{N_0} \Psi^{k\ell} \int_R \nabla \phi_{ij}(x,y) \cdot \nabla \phi_{k\ell}(x,y)\, \mathrm{d}x\, \mathrm{d}y = g(x_i, y_j) \int_R \phi_{ij}(x,y)\, \mathrm{d}x\, \mathrm{d}y \qquad (1)$$

- N coupled inhomogeneous linear algebraic equations for the N unknowns, $\Psi^{k\ell}$.

- The coupling among $\Psi^{k\ell}$ is confined to a small number of sites near (x_i,y_j) for the localized function ϕ_{ij}.

$$\int_R \phi_{ij}(x,y)\, \mathrm{d}x\, \mathrm{d}y = h^2$$

$$\int_R \nabla \phi_{ij} \cdot \nabla \phi_{k\ell}\, \mathrm{d}x\, \mathrm{d}y = \begin{cases} 8/3 & \text{for} \quad k=i,\ \ell=j \\ -1/3 & \text{for} \quad \left[\begin{array}{l} k=i\pm 1,\ \ell=j \\ k=i,\ \quad \ell=j\pm 1 \\ k=i\pm 1,\ \ell=j\pm 1 \end{array}\right. \end{cases}$$

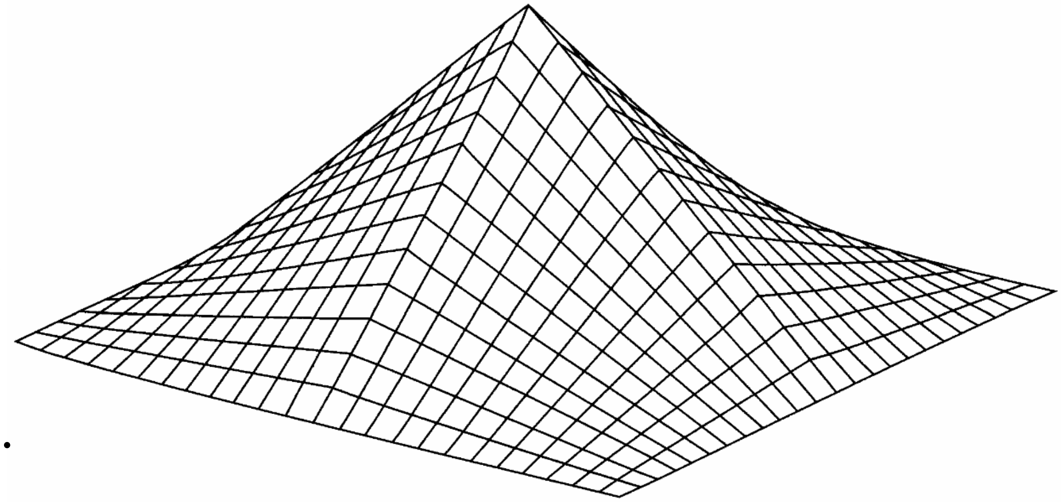


- Eq (1) can be written in matrix form:

$$\mathbf{K}\,\boldsymbol{\Psi} = \mathbf{G}$$

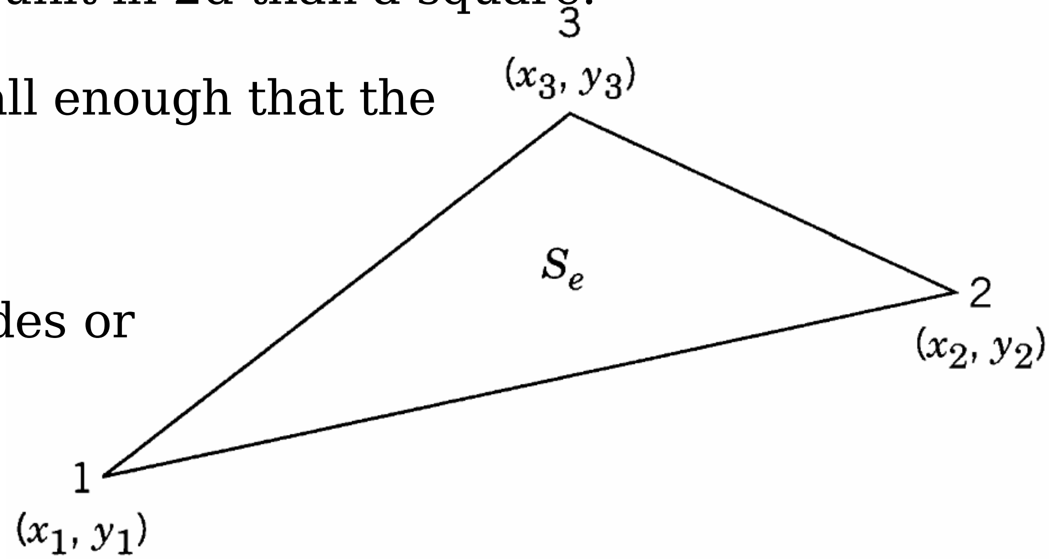
$\mathbf{K}: N \times N$ matrix

$\boldsymbol{\Psi}$ and $\mathbf{G}: N$ -column vectors



- Since **K** is a sparse matrix, the solution can be rapidly found by special numerical techniques.

- A triangle is more optimal as a basic unit in 2d than a square.



- The triangular element shoud be small enough that the field variable change little such that

$$\psi(x, y) \approx \psi_e(x, y) = A + B\,x + C\,y.$$

then use the value (ψ_1, ψ_2, ψ_3) at nodes or vertices to determine (A,B,C).

- To be systematic, define 3 shape functions $N_j^e(x,y)$ at the vertices such that $N_j^e(x = x_j, y = y_j) = 1$ only.

- Consider $N_1^e = a_1 + b_1\,x + c_1\,y \quad \Rightarrow$

$$a_1 + b_1\,x_1 + c_1\,y_1 = 1$$

$$a_1 + b_1\,x_2 + c_1\,y_2 = 0$$

$$a_1 + b_1\,x_3 + c_1\,y_3 = 0$$

- The determinant $D = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) = \overrightarrow{12} \times \overrightarrow{13}$

- D is invariant under rotations of the triangle, and

$$D = 2S_e \quad \Leftarrow \quad S_e : \text{area of the triangle} \quad \Rightarrow \quad \begin{array}{l} a_1 = \dfrac{x_2 y_3 - x_3 y_2}{2 S_e}, \quad b_1 = \dfrac{y_2 - y_3}{2 S_e} \\ c_1 = \dfrac{x_3 - x_2}{2 S_e} \end{array}$$

- The other N_j^e can be known by cyclic permutation of indices.

$$\sum_{i=1}^{3} N_i^e(x, y) = 1, \quad \sum_{i=1}^{3} a_i = 1, \quad \sum_{i=1}^{3} b_i = 0, \quad \sum_{i=1}^{3} c_i = 0$$

$$a_j + b_j \bar{x}_e + c_j \bar{y}_e = \frac{1}{3} \quad (j = 1, 2, 3) \quad \text{where} \quad \bar{x}_e = \frac{x_1 + x_2 + x_3}{3}, \quad \bar{y}_e = \frac{y_1 + y_2 + y_3}{3}$$

- The field variable $\psi(x,y)$ has the expansion: $\psi(x, y) \approx \sum_{f,j} \Psi_j^f N_j^f(x, y)$

  where f: triangle number

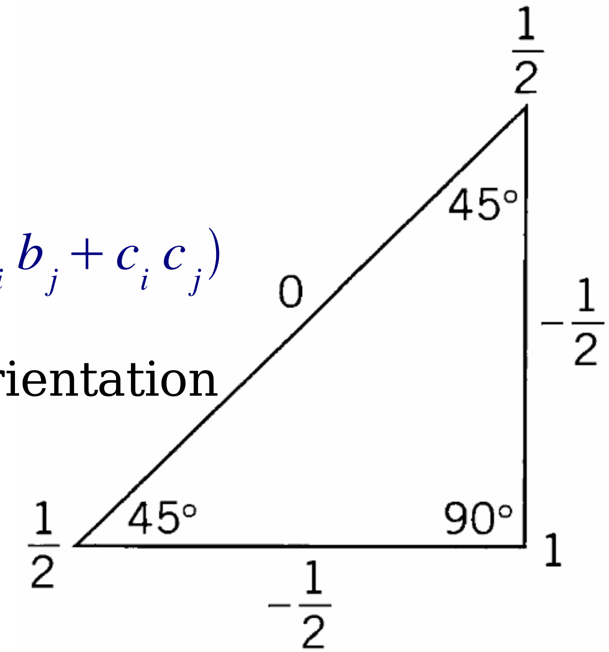
- Because of the linearity of the shape functions, the value along the common side of the two triangles from either representation is the same weighted average of the values at each end.

- choose the test function $\phi(x,y)=N_j^e(x,y)$ for some element e and vertex i, then

$$\sum_{j=1}^{3} \Psi_j^e \int_e \nabla N_i^e \cdot \nabla N_j^e \, \mathrm{d}x\,\mathrm{d}y = \int_e g N_i^e \, \mathrm{d}x\,\mathrm{d}y \approx g(\bar{x}_e, \bar{y}_e) \int_e N_i^e \, \mathrm{d}x\,\mathrm{d}y \qquad (2)$$

$$\int_e N_i^e \, \mathrm{d}x\,\mathrm{d}y = S_e (a_i + b_i \bar{x}_e + c_1 \bar{y}_e) = S_e/3$$

$$\frac{\partial N_i^e}{\partial x} = b_i, \quad \frac{\partial N_i^e}{\partial y} = c_i \quad \Rightarrow k_{ij}^e \equiv \int_e \nabla N_i^e \cdot \nabla N_j^e \, \mathrm{d}x\,\mathrm{d}y = S_e (b_i b_j + c_i c_j)$$



- K^e_{ij}'s only depend on the shape of triangle, not in the orientation or size.

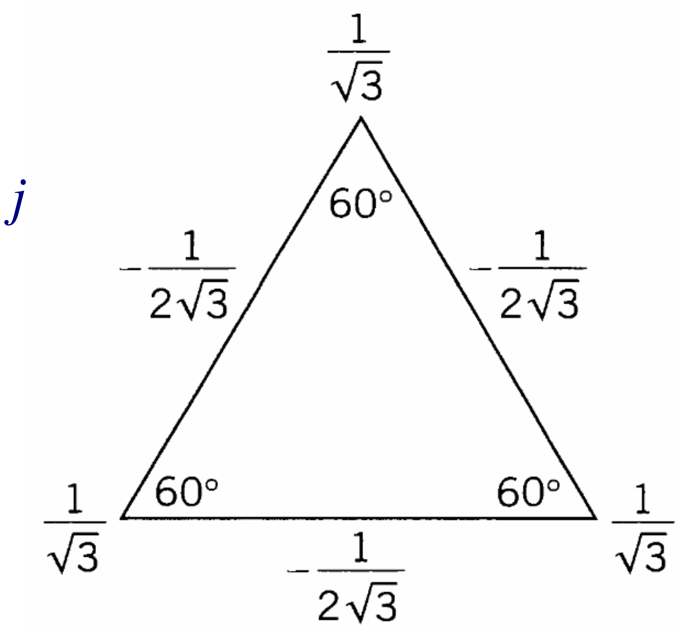
$$(2) \Rightarrow \sum_{j=1}^{3} k_{ij}^e \Psi_j^e = \frac{S_e g_e}{3} \quad (i=1,2,3) \quad \Leftarrow \quad g_e \equiv g(\bar{x}_e, \bar{y}_e)$$

$$\mathbf{K}^e \boldsymbol{\Psi}^e = \mathbf{G}^e \quad \text{in matrix form}$$

- Enlarge the matrix to include all triangles: $\mathbf{K}\,\boldsymbol{\Psi} = \mathbf{G}$

where $\mathbf{K} = \|k_{ij}\|, \quad \Leftarrow \quad k_{ii} = \sum_T k_{ii}^e, \quad k_{ij} = \sum_E k_{ij}^e \quad \text{for} \quad i \neq j$

$$G_i = \frac{1}{3} \sum_T S_e g_e - \sum_{j=N+1}^{N_0} k_{ij}^e \Psi_j^e \quad \Leftarrow (\text{boundary terms})$$



T: all the triangles connected to the node

E: all the triangles with a side from node i to node j

- The obvious generalization of the triangle to 3d FEA is to add another vertex out of the plane to make a tetrahedron the basic element of volume.

- Selected problems:
2.5, 2.7, 2.13, 2.15, 2.20, 2.26