Chapter 1 Introduction to Electrostatics

- *Electrostatics* — phenomena involving time-independent distributions of charge and fields.

- Electrostatics developed as a science of *macroscopic* phenomena. Some idealizations like point charge may fail to have meaning microscopically.

1.1 Coulomb's Law

- the force between two small charged bodies separated in a distance

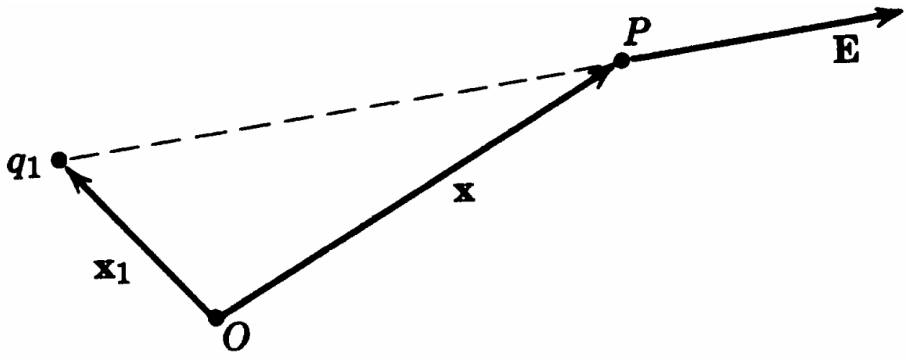
  - varies directly as the magnitude of each charge,
  - varies inversely as the square of the distance,
  - is directed along the line joining the charges,
  - attractive if oppositely charged and repulsive if the same type of charge,
  - the vector summation rule applies.

1.2 Electric Field

- Electric field: force/(unit charge) at a given point in a limiting process $\mathbf{F} = q\,\mathbf{E}$

- Coulomb's Law: $\mathbf{F}(\mathbf{x}_1, \mathbf{x}_2) = k\,q_1\,q_2\,\dfrac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3}$

- the electric field: $\mathbf{E}(\mathbf{x}) = k\,q_1\,\dfrac{\mathbf{x} - \mathbf{x}_1}{|\mathbf{x} - \mathbf{x}_1|^3}$



- In the SI system: $k = (4\pi\epsilon_0)^{-1} = 10^{-7}\, c^2$ $\Leftarrow$ free space permittivity $\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{F/M}$
- The linear superposition law:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} q_i \frac{\mathbf{x} - \mathbf{x}_1}{|\mathbf{x} - \mathbf{x}_1|^3} \quad \Rightarrow \quad \mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \mathrm{d}^3 x'$$

where $\Delta q = \rho(\mathbf{x}')\, \Delta x\, \Delta y\, \Delta z$, $\mathrm{d}^3 x' = \mathrm{d}x'\, \mathrm{d}y'\, \mathrm{d}z'$

- **Dirac delta function**: a mathematically improper function with the properties
  1. $\delta(x-a) = 0$ for $x \neq a$ in 1d,
  2. $\int \delta(x-a)\, \mathrm{d}x = 1$ if the region of integration includes $x = a$, and is zero otherwise
  3. $\int f(x)\, \delta(x-a)\, \mathrm{d}x = f(a)$
  4. $\int f(x)\, \delta'(x-a)\, \mathrm{d}x = -f'(a)$
  5. $\delta(f(x)) = \sum_i \left| \frac{\mathrm{d}f}{\mathrm{d}x}(x_i) \right|^{-1} \delta(x - x_i)$
  6. $\delta(\mathbf{x} - \mathbf{X}) = \delta(x_1 - X_1)\, \delta(x_2 - X_2)\, \delta(x_3 - X_3)$ with Cartesian coordinates in 3d
  7. $\int_{\Delta V} \delta(\mathbf{x} - \mathbf{X})\, \mathrm{d}^3 x = \begin{cases} 1 & \text{if } \Delta V \text{ constains } \mathbf{x} = \mathbf{X} \\ 0 & \text{if } \Delta V \text{ does not constain } \mathbf{x} = \mathbf{X} \end{cases} \quad \Rightarrow \quad [\delta(\mathbf{x} - \mathbf{X})] = \frac{1}{V}$
- A discrete set of point charges can be described with delta functions

$$\rho(\mathbf{x}) = \sum_{i=1}^{n} q_i\, \delta(\mathbf{x} - \mathbf{x}_i)$$

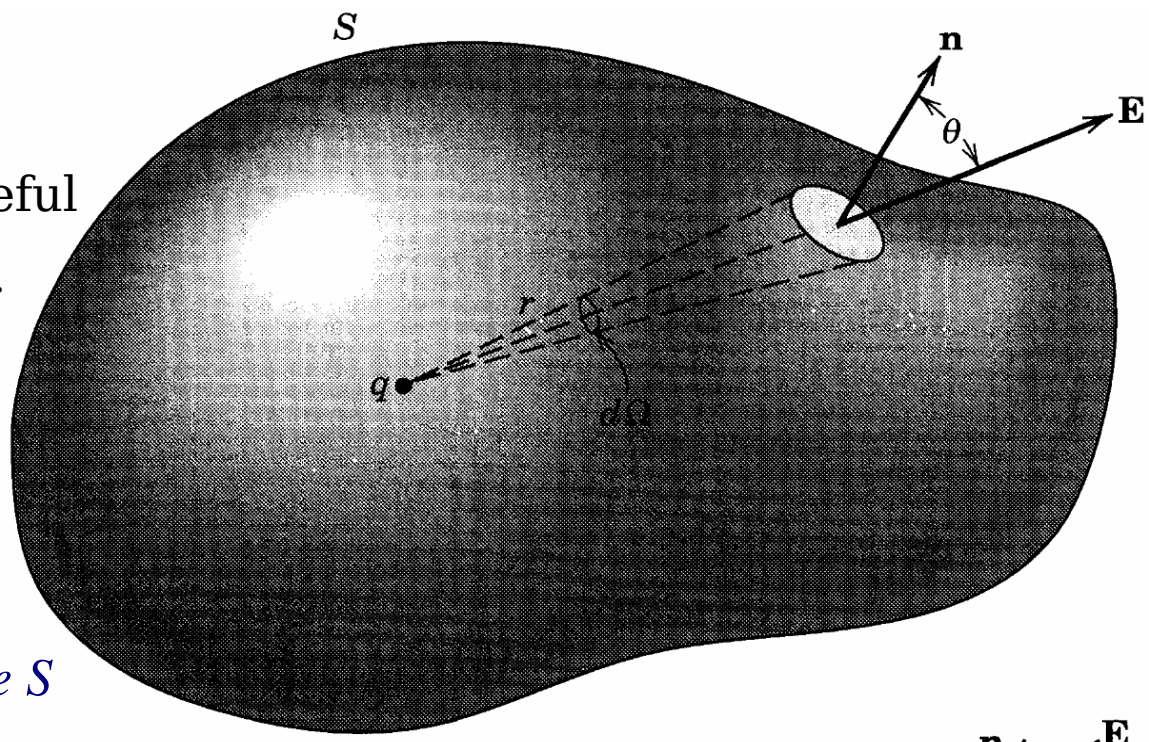
1.3 Gauss's Law

- *Gauss's law* is sometimes more useful and leads to a differential eqn for E.

$$\mathbf{E}\cdot\mathbf{n}\,\mathrm{d}a = \frac{q}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}\,\mathrm{d}a$$

$$= \frac{q}{4\pi\epsilon_0}\,\mathrm{d}\Omega \quad \Leftarrow \quad r^2\,\mathrm{d}\Omega = \cos\theta\,\mathrm{d}a$$

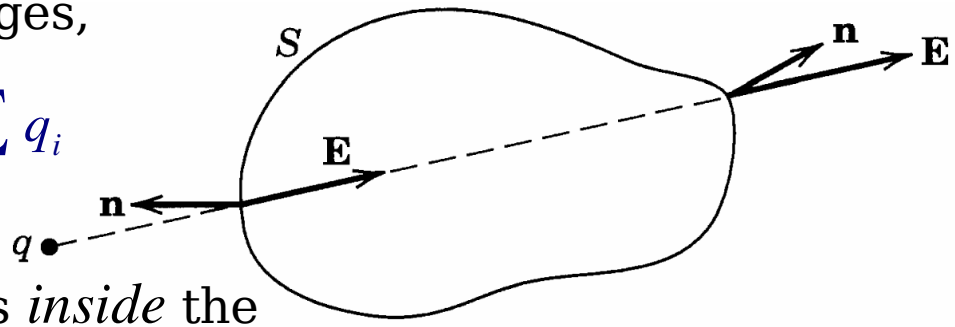
$$\Rightarrow \quad \oint_S \mathbf{E}\cdot\mathbf{n}\,\mathrm{d}a = \begin{cases} q/\epsilon_0 & \text{if } q \text{ lies } \textit{inside } S \\ 0 & \text{if } q \text{ lies } \textit{outside } S \end{cases}$$



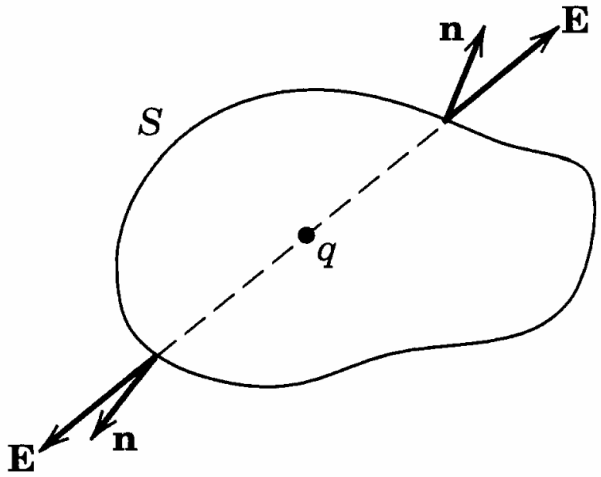
- For a set of charges,

$$\oint_S \mathbf{E}\cdot\mathbf{n}\,\mathrm{d}a = \frac{1}{\epsilon_0}\sum_i q_i$$

the sum is over only those charges *inside* the surface S.



q outside S



q inside S

- For a continuous charge density: $\oint_S \mathbf{E}\cdot\mathbf{n}\,\mathrm{d}a = \frac{1}{\epsilon_0}\int_V \rho(\mathbf{x})\,\mathrm{d}^3x$

- The eqn is one of the basic eqns of electrostatics. It depends upon (a) the inverse square law for the force between charges; (b) the central nature of the force; (c) the linear superposition of the effects of different charges.

1.4 Differential Form of Gauss's Law

- The *divergence theorem*: for any well-behaved vector field defined within a volume surrounded by the closed surface $\oint_S \mathbf{A}\cdot\mathbf{n}\,\mathrm{d}\,a = \int_V \nabla\cdot\mathbf{A}\,\mathrm{d}^3\,x$

- apply the divergence theorem $\oint_S \mathbf{E}\cdot\mathbf{n}\,\mathrm{d}\,a = \int_V \nabla\cdot\mathbf{E}\,\mathrm{d}^3\,x = \frac{1}{\epsilon_0}\int_V \rho\,(\mathbf{x})\,\mathrm{d}^3\,x$

$\Rightarrow \quad \int_V (\nabla\cdot\mathbf{E} - \rho/\epsilon_0)\,\mathrm{d}^3\,x$ for an arbitrary volume V $\Rightarrow$ $\boxed{\nabla\cdot\mathbf{E} - \frac{\rho}{\epsilon_0} = 0}$

the differential form of Gauss's law of electrostatics.

1.5 Another Equation of Electrostatics and the Scalar Potential

- A vector field can be specified almost completely if its divergence and curl are given everywhere in space.

- $\nabla \times \nabla \psi = 0 \quad \text{for all} \quad \psi$

- look for an equation specifying curl E as a function of position,

$$\nabla \times \mathbf{E} = 0 \quad \Leftarrow \quad \mathbf{E}(\mathbf{x}) = -\frac{1}{4\pi\epsilon_0} \nabla \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \mathrm{d}^3 x' \Leftarrow \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{x}') \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \mathrm{d}^3 x'$$

- Note that $\nabla \times \mathbf{E} = 0$ depends on the central nature of the force, and on the fact that the force is a function of relative distances only, but does not depend on the inverse square nature.

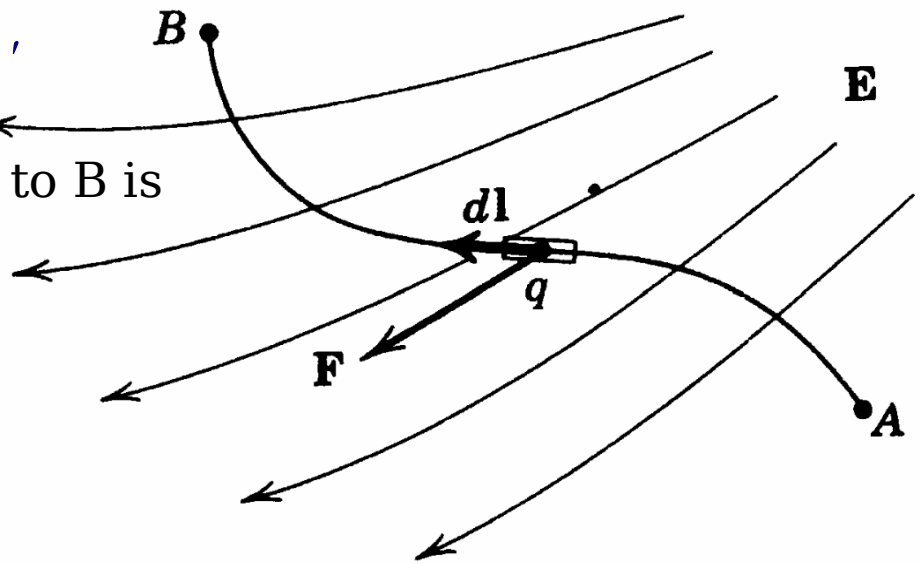
- Since a scalar is easier to deal with than a vector, define the *scalar potential*

$$\mathbf{E} = -\nabla \Phi \quad \Rightarrow \quad \Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \mathrm{d}^3 x'$$

- the work done in moving the charge from A to B is

$$W = -\int_A^B \mathbf{F} \cdot \mathrm{d}\boldsymbol{\ell} = -q \int_A^B \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} \quad \Leftarrow \quad \mathbf{F} = q\,\mathbf{E}$$

$$= q \int_A^B \nabla \Phi \cdot \mathrm{d}\boldsymbol{\ell} = q \int_A^B \mathrm{d}\Phi = q(\Phi_B - \Phi_A)$$



- $q\Phi$ can be interpreted as the potential energy of the test charge in the electrostatic field.

- the line integral of the electric field between two points is independent of the path and is the negative of the potential difference between the points:

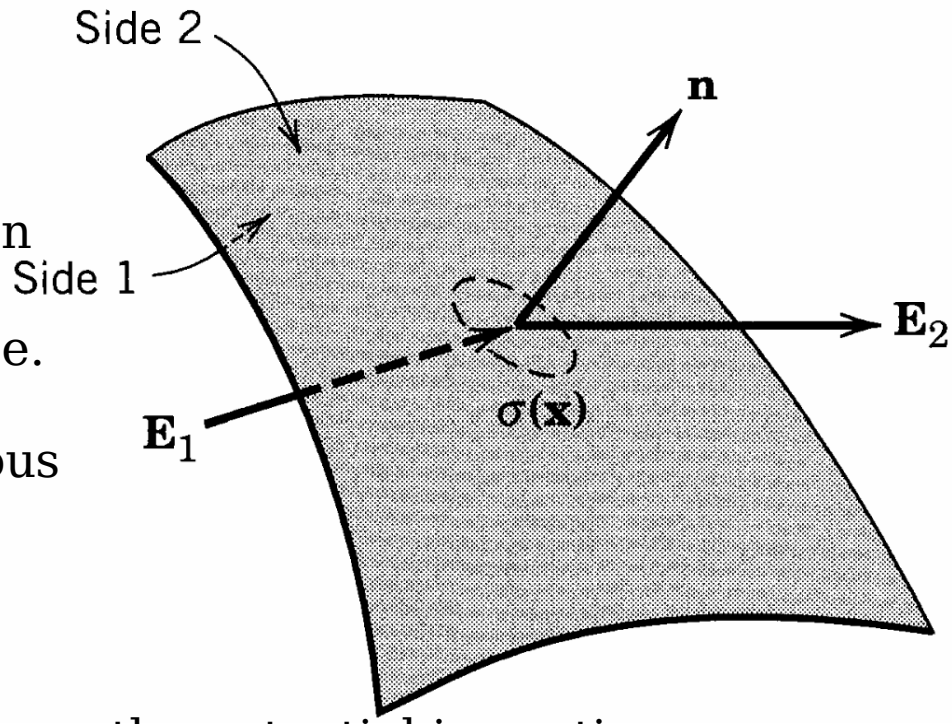
$$\int_A^B \mathbf{E}\cdot \mathrm{d}\,\boldsymbol{\ell} = -(\Phi_B - \Phi_A) \quad \Rightarrow \quad \oint \mathbf{E}\cdot \mathrm{d}\,\boldsymbol{\ell} = 0 \quad (\star) \quad \Rightarrow$$

If the path is closed, the line integral is zero

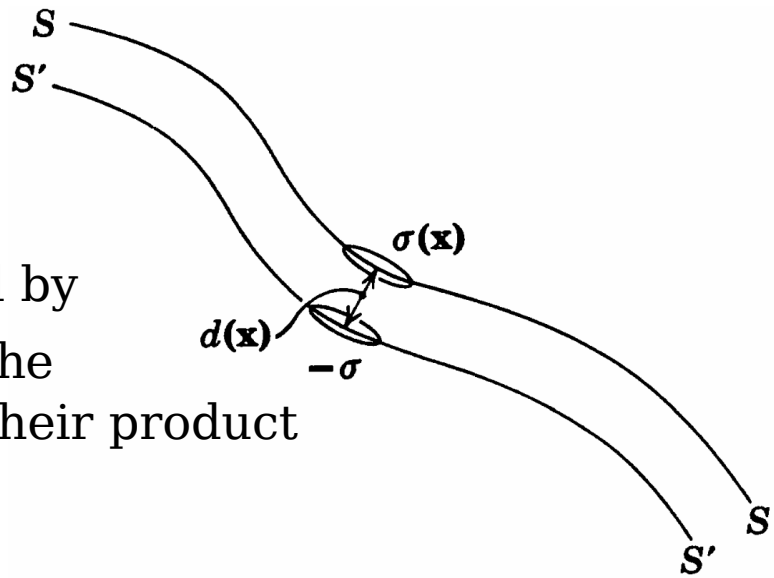
- *Stokes's theorem*: $\oint_C \mathbf{A}\cdot \mathrm{d}\,\boldsymbol{\ell} = \int_S (\nabla \times \mathbf{A})\cdot \mathbf{n}\,\mathrm{d}\,a$

- With the line integral of the electric field being independent of the path and the application of the Stokes's theorem $\Rightarrow \quad \nabla \times \mathbf{E} = 0$

1.6 Surface Distributions of Charges & Dipoles and Discontinuities in the Electric Field & Potential



- Gauss's law tells $(\mathbf{E}_2 - \mathbf{E}_1)\cdot\mathbf{n} = \dfrac{\sigma}{\epsilon_0}$
- This does not determine $\mathbf{E}_1$ and $\mathbf{E}_2$. The eqn means that there is a discontinuity in the normal component of $\mathbf{E}$ in crossing a surface.
- The tangential component of $\mathbf{E}$ is continuous across a boundary surface from eqn (*).
- In this case $\Phi(\mathbf{x}) = \dfrac{1}{4\pi\epsilon_0}\int_S \dfrac{\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}a'$
- For volume or surface distributions of charge, the potential is continuous, even within the charge distribution. $\mathbf{E}$ is bounded, even though discontinuous across a surface distribution of charge.



- With point or line charges, or dipole layers, the potential is no longer continuous.
- The dipole-layer distribution of strength is formed by letting S' approach infinitesimally close to S while the surface-charge density becomes infinite such that their product approaches the limit:

$$\lim_{d(\mathbf{x})\to 0} \sigma(\mathbf{x})\, d(\mathbf{x}) = D(\mathbf{x})$$

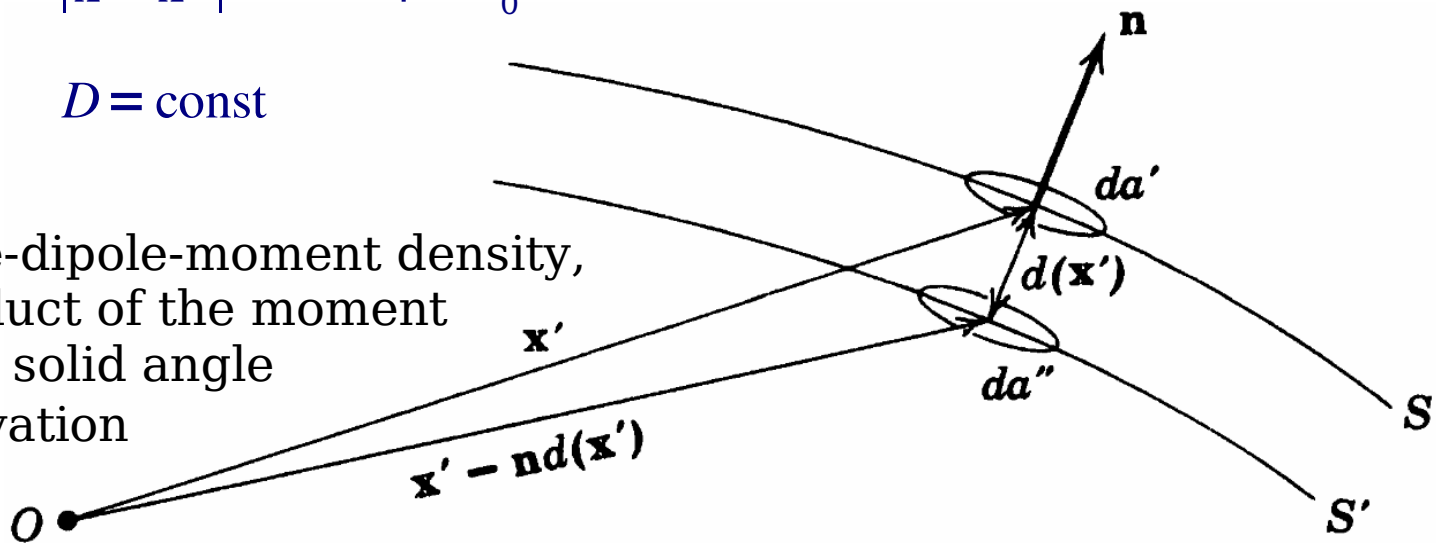
- $$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}a' - \frac{1}{4\pi\epsilon_0}\int_{S'} \frac{\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'+\mathbf{n}d|}\,\mathrm{d}a''$$

$$\approx \frac{1}{4\pi\epsilon_0}\int_S D(\mathbf{x}')\,\mathbf{n}\cdot\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\mathrm{d}a' \quad \Leftarrow \quad \frac{1}{|\mathbf{x}+\mathbf{a}|} = \frac{1}{x} + \mathbf{a}\cdot\nabla\left(\frac{1}{x}\right) + \cdots$$

$$= -\frac{1}{4\pi\epsilon_0}\int_S D(\mathbf{x}')\frac{\cos\theta\,\mathrm{d}a'}{|\mathbf{x}-\mathbf{x}'|^2} = -\frac{1}{4\pi\epsilon_0}\int_S D(\mathbf{x}')\,\mathrm{d}\Omega$$

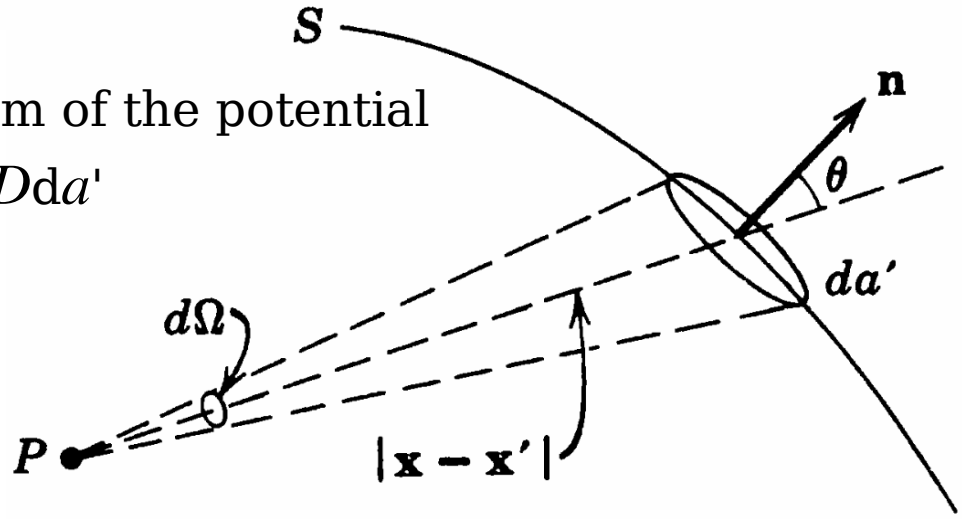
$$\to -\frac{\Delta\Omega}{4\pi\epsilon_0}D \quad \text{if} \quad D = \text{const}$$

- For a constant surface-dipole-moment density, the potential is the product of the moment divided by $4\pi\epsilon_0$ and the solid angle subtended at the observation point by the surface, regardless of its shape.



- In the integrand we notice that it is the sum of the potential of a point dipole with dipole moment $\mathrm{d}\mathbf{p} = \mathbf{n}D\mathrm{d}a'$

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{(\mathbf{x}-\mathbf{x}')\cdot\mathrm{d}\,\mathbf{p}}{|\mathbf{x}-\mathbf{x}'|^3}$$



- There is a discontinuity in potential in crossing a double layer. The total potential jump in crossing the surface is:

$$\Phi_2 - \Phi_1 = \frac{D}{\epsilon_0}$$

- The potential has a discontinuity of D/ϵ_0 in crossing from the inner to the outer side, being $-D/2\epsilon_0$ on the inner side and $+D/2\epsilon_0$ on the outer.

1.7 Poisson and Laplace Equations

- $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0} \quad \Rightarrow \quad \nabla^2 \Phi = -\dfrac{\rho}{\epsilon_0} \quad \Leftarrow$ Poisson equation

  $\nabla \times \mathbf{E} = 0 \quad \Rightarrow \quad \mathbf{E} = -\nabla \Phi \qquad \nabla^2 \Phi = 0 \quad \Leftarrow$ Laplace equation

$$\Rightarrow \quad \Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \, \mathrm{d}^3 x'$$

- To verify the result directly and avoid being singular in the resulting integrand, we invoke a limiting procedure:

$$\Phi_a(\mathbf{x}) \equiv \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}')}{\sqrt{(\mathbf{x}-\mathbf{x}')^2 + a^2}} \, \mathrm{d}^3 x', \qquad r \equiv |\mathbf{x}-\mathbf{x}'|$$

$$\Rightarrow \quad \nabla^2 \Phi_a(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{x}') \nabla^2 \frac{1}{\sqrt{r^2 + a^2}} \, \mathrm{d}^3 x' = -\frac{1}{4\pi\epsilon_0} \int \frac{3a^2 \rho(\mathbf{x}')}{(r^2 + a^2)^{5/2}} \, \mathrm{d}^3 x'$$

- It is well-behaved everywhere for nonvanishing a, but as $a \to 0$ $\begin{cases} \Phi \to \infty, & r = 0 \\ \Phi = 0, & r \neq 0 \end{cases}$

$$\nabla^2 \Phi_a(r \to 0) \approx -\frac{3a^2 \rho(\mathbf{x})}{4\pi\epsilon_0} \int \frac{\mathrm{d}^3 x}{(r^2 + a^2)^{5/2}} = -\frac{3a^2 \rho(\mathbf{x})}{\epsilon_0} \int_0^\infty \frac{r^2 \, \mathrm{d}r}{(r^2 + a^2)^{5/2}} = -\frac{\rho(\mathbf{x})}{\epsilon_0}$$

- The singular nature of $\nabla^2(1/r)$ can be expressed as $\boxed{\nabla^2 \dfrac{1}{|\mathbf{x}-\mathbf{x}'|} = -4\pi\delta(\mathbf{x}-\mathbf{x}')}$

1.8 Green's Theorem

- To handle the boundary conditions it is necessary to develop some new mathematical tools. The divergence theorem:

$$\int_V \nabla\cdot\mathbf{A}\,\mathrm{d}^3x=\oint_S \mathbf{A}\cdot\mathbf{n}\,\mathrm{d}a \quad \text{and} \quad \mathbf{A}=\phi\nabla\psi \quad \Rightarrow \quad \begin{cases}\nabla\cdot(\phi\nabla\psi)=\phi\nabla^2\psi+\nabla\phi\cdot\nabla\psi\\ \phi\nabla\psi\cdot\mathbf{n}=\phi\dfrac{\partial\psi}{\partial n}\end{cases}$$

$$\Rightarrow \quad \boxed{\int_V(\phi\nabla^2\psi+\nabla\phi\cdot\nabla\psi)\,\mathrm{d}^3x=\oint_S \phi\frac{\partial\psi}{\partial n}\,\mathrm{d}a} \quad \Leftarrow \quad (\text{Green's 1st identity}) \quad \Rightarrow \quad \phi\leftrightarrow\psi$$

$$\Rightarrow \quad \boxed{\int_V(\phi\nabla^2\psi-\psi\nabla^2\phi)\,\mathrm{d}^3x=\oint_S\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)\mathrm{d}a} \quad \Leftarrow \quad \begin{pmatrix}\text{Green's 2nd identiy}\\ \text{Green theorem}\end{pmatrix}$$

- Convert the Poisson differential equation into an integral equation

$$\nabla^2\Phi=-\frac{\rho}{\epsilon_0}, \quad \psi=\frac{1}{R}=\frac{1}{|\mathbf{x}-\mathbf{x}'|} \quad \Rightarrow \quad \nabla^2\psi=-4\pi\delta(\mathbf{x}-\mathbf{x}')$$

$$\Rightarrow \quad \int_V\left[-4\pi\Phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')+\frac{\rho(\mathbf{x}')}{\epsilon_0 R}\right]\mathrm{d}^3x=\oint_S\left(\Phi\frac{\partial}{\partial n'}\frac{1}{R}-\frac{1}{R}\frac{\partial\Phi}{\partial n'}\right)\mathrm{d}a'$$

$$\Rightarrow \quad \Phi(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\int_V\frac{\rho(\mathbf{x}')}{R}\mathrm{d}^3x+\frac{1}{4\pi}\oint_S\left(\frac{1}{R}\frac{\partial\Phi}{\partial n'}-\Phi\frac{\partial}{\partial n'}\frac{1}{R}\right)\mathrm{d}a' \quad \Leftarrow \quad \mathbf{x}\in V \quad (\star')$$

$$0=\frac{1}{4\pi\epsilon_0}\int_V\frac{\rho(\mathbf{x}')}{R}\mathrm{d}^3x+\frac{1}{4\pi}\oint_S\left(\frac{1}{R}\frac{\partial\Phi}{\partial n'}-\Phi\frac{\partial}{\partial n'}\frac{1}{R}\right)\mathrm{d}a' \quad \Leftarrow \quad \mathbf{x}\notin V$$

- For eqn (★') if $S \to \infty$, $E(S) \propto R^{-1+\Delta} \quad \Rightarrow \quad \Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \mathrm{d}^3 x'$
- $\rho(\mathbf{x}') = 0 \quad \Rightarrow \quad \Phi(\mathbf{x}) = \frac{1}{4\pi} \oint_S \left(\frac{1}{R} \frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'} \frac{1}{R} \right) \mathrm{d}a' = \oint_S f\left(\Phi, \frac{\partial \Phi}{\partial n'}\right) \mathrm{d}a'$

This result is not a solution to a boundary-value problem, but only an integral statement, since the arbitrary specification of both Φ and Φ' (Cauchy boundary conditions) is an overspecification of the problem.

1.9 Uniqueness of the Solution with Dirichlet or Neumann Boundary Conditions

- *Dirichlet problem/Dirichlet boundary condition*: specification of the potential on a closed surface defines a unique potential problem.

- *Neumann boundary condition*:specification of the electric field (normal derivative of the potential) everywhere on the surface also defines a unique problem.

- to show the uniqueness of the solution of the Poisson equation inside a volume subject to either Dirichlet or Neumann boundary conditions on the closed bounding surface.

- Assume 2 solutions satisfy the same boundary conditions and $U = \Phi_1 - \Phi_2$

$$\Rightarrow \quad \nabla^2 U = 0 \quad \text{in} \quad V, \quad \text{and} \quad U\big|_S = 0 \ (\text{Dirichlet}), \quad \text{or} \quad \left.\frac{\partial U}{\partial n}\right|_S = 0 \ (\text{Neumann})$$

$$\Rightarrow \quad \int_V (U \nabla^2 U + \nabla U \cdot \nabla U)\, \mathrm{d}^3 x = \oint_S U \frac{\partial U}{\partial n}\, \mathrm{d}\, a \quad \Leftarrow \phi = \psi = U \quad \Rightarrow \int_V |\nabla U|^2\, \mathrm{d}^3 x = 0$$

$$\Rightarrow \quad \nabla U = 0 \ \text{inside} \ V \quad \Rightarrow \quad U = \text{const} \quad \Rightarrow \quad \begin{array}{l} U = 0 \quad (\text{Dirichlet}) \quad \Rightarrow \quad \Phi_1 = \Phi_2 \\ U = \text{const} \ (\text{Neumann}) \quad \Rightarrow \quad \Phi_1 = \Phi_2 + \text{const} \end{array}$$

- there is also a unique solution to a problem with mixed boundary conditions.

- A solution to the Poisson equation doesn't necessarily exist with arbitrary Φ & Φ'

1.10 Formal Solution of Electrostatic Boundary-Value Problem with Green Function

- $\nabla'^2 |\mathbf{x}-\mathbf{x}'|^{-1} = -4\pi\,\delta(\mathbf{x}-\mathbf{x}') \quad \Leftarrow \quad$ the potential of a unit point source

- The function is only one of a class of functions depending on the variables **x** and **x'**, and called *Green functions*, which satisfy

$$\nabla'^2 G(\mathbf{x},\mathbf{x}') = -4\pi\,\delta(\mathbf{x}-\mathbf{x}') \quad \Leftarrow \quad G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + F(\mathbf{x},\mathbf{x}') \quad \Rightarrow \quad \nabla'^2 F(\mathbf{x},\mathbf{x}') = 0$$

- With the generalized concept of a Green function and its additional freedom $F(\mathbf{x}, \mathbf{x}')$, there arises the possibility that we can use Green's theorem and choose $F(\mathbf{x}, \mathbf{x}')$ to eliminate one or the other of the two surface integrals, obtaining a result that involves only Dirichlet or Neumann boundary conditions.

- $\phi = \Phi$, $\psi = G(\mathbf{x},\mathbf{x}')$ $\Rightarrow$

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')\,G(\mathbf{x},\mathbf{x}')\,\mathrm{d}^3x' + \frac{1}{4\pi}\oint_S \left[G(\mathbf{x},\mathbf{x}')\frac{\partial\Phi}{\partial n'} - \Phi(\mathbf{x}')\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'}\right]\mathrm{d}a'$$

- for *Dirichlet boundary conditions* we demand: $G_D(\mathbf{x},\mathbf{x}') = 0 \quad \text{for} \quad \mathbf{x}' \quad \text{on} \; S$

$$\Rightarrow \quad \boxed{\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')\,G_D(\mathbf{x},\mathbf{x}')\,\mathrm{d}^3x' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial G_D}{\partial n'}\,\mathrm{d}a'}$$

- Gauss's theorem gives $-4\pi = \int_V \nabla^2 G \, \mathrm{d}^3 x' = \oint_S \nabla G \cdot \mathrm{d}\mathbf{a}' = \oint_S \frac{\partial G}{\partial n'} \, \mathrm{d}a'$
- For Neumann boundary conditions, the simplest allowable one is

$$\frac{\partial G_N}{\partial n'}(\mathbf{x}, \mathbf{x}') = -\frac{4\pi}{S} \quad \text{for} \quad \mathbf{x}' \quad \text{on } S \quad \text{instead of} \quad \frac{\partial G_N}{\partial n'}(\mathbf{x}, \mathbf{x}') = 0 \quad \text{for} \quad \mathbf{x}' \quad \text{on } S$$

where S is the total area of the boundary surface

$$\Rightarrow \quad \boxed{\Phi(\mathbf{x}) = \langle \Phi \rangle_S + \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{x}') G_N(\mathbf{x}, \mathbf{x}') \, \mathrm{d}^3 x' + \frac{1}{4\pi} \oint_S G_N \frac{\partial \Phi}{\partial n'} \, \mathrm{d}a'}$$

- The customary Neumann problem is the so-called exterior problem in which the volume is bounded by two surfaces, one closed and finite, the other at infinity. Then the surface area is infinite; the average value vanishes.

- The mathematical symmetry property $G(\mathbf{x}, \mathbf{x}') = G(\mathbf{x}', \mathbf{x})$ merely represents the physical interchangeability of the source and the observation points.

- For the physical meaning of $F(\mathbf{x}, \mathbf{x}')/4\pi\epsilon_0$, it is a solution of the Laplace equation inside V and represents the potential of charges *external to the volume V*.

1.11 Electrostatic Potential Energy & Energy Density; Capacitance

- If a point charge is brought from infinity to a point in a scalar potential (which vanishes at infinity), the work done on the charge is

$$W_i = q_i \Phi(\mathbf{x}_i) = \frac{q_i}{4\pi\epsilon_0} \sum_{j=1}^{n-1} \frac{q_j}{|\mathbf{x}_i - \mathbf{x}_j|} \quad \Leftarrow \quad \Phi(\mathbf{x}_i) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{n-1} \frac{q_j}{|\mathbf{x}_i - \mathbf{x}_j|}$$

$$\Rightarrow \quad W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j<i} \frac{q_i q_j}{|\mathbf{x}_i - \mathbf{x}_j|} = \frac{1}{8\pi\epsilon_0} \sum_i \sum_j \frac{q_i q_j}{|\mathbf{x}_i - \mathbf{x}_j|} \quad \text{for} \quad i \neq j \quad (\text{no self-energy term})$$

$$\Rightarrow \quad W = \frac{1}{8\pi\epsilon_0} \int\int \frac{\rho(\mathbf{x})\,\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \mathrm{d}^3x\, \mathrm{d}^3x' = \frac{1}{2} \int \rho(\mathbf{x})\,\Phi(\mathbf{x})\,\mathrm{d}^3x \quad \text{for a continuous charge distribution}$$

- expresses the electrostatic potential energy in terms of the positions of the charges and emphasize the interactions between charges via Coulomb forces.

- An alternative approach is to emphasize the electric field and to interpret the energy as being stored in the electric field surrounding the charges

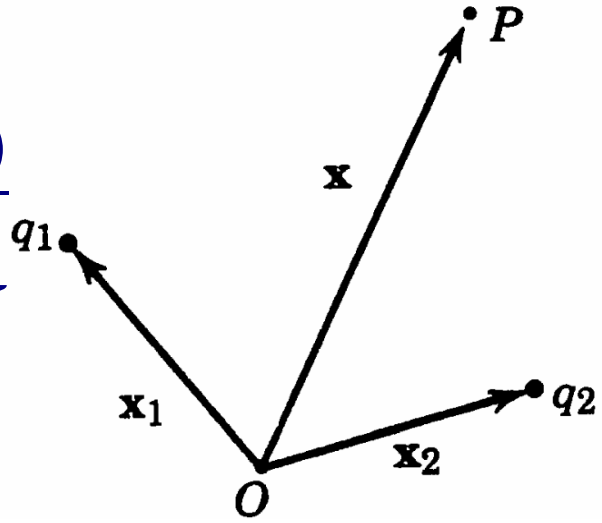
$$\Rightarrow \quad W = \frac{1}{2} \int \rho(\mathbf{x})\,\Phi(\mathbf{x})\,\mathrm{d}^3x = -\frac{\epsilon_0}{2} \int \Phi\,\nabla^2\Phi\,\mathrm{d}^3x = \frac{\epsilon_0}{2} \int |\nabla\Phi|^2\,\mathrm{d}^3x = \frac{\epsilon_0}{2} \int |\mathbf{E}|^2\,\mathrm{d}^3x$$

$$\Rightarrow \quad w = \frac{\epsilon_0}{2} |\mathbf{E}|^2 \geq 0 \quad \Leftarrow \quad \text{energy density}$$

- The energy density is positive definite. This contradicts our impression that the potential energy of two charges of opposite sign is negative. This apparent contradiction comes from "self-energy" contributions to the energy density. Ex:

$$\mathbf{E} = \frac{q_1}{4\pi\epsilon_0}\frac{\mathbf{x}-\mathbf{x_1}}{|\mathbf{x}-\mathbf{x_1}|^3} + \frac{q_2}{4\pi\epsilon_0}\frac{\mathbf{x}-\mathbf{x_2}}{|\mathbf{x}-\mathbf{x_2}|^3}$$

$$\Rightarrow \quad 32\pi^2\epsilon_0 w = \underbrace{\frac{q_1^2}{|\mathbf{x}-\mathbf{x_1}|^4} + \frac{q_2^2}{|\mathbf{x}-\mathbf{x_2}|^4}}_{\text{self energy}} + 2\underbrace{\frac{q_1 q_2 (\mathbf{x}-\mathbf{x_1})\cdot(\mathbf{x}-\mathbf{x_2})}{|\mathbf{x}-\mathbf{x_1}|^3|\mathbf{x}-\mathbf{x_2}|^3}}_{w_{\text{int}}}$$



define $\quad \boldsymbol{\rho} \equiv \frac{\mathbf{x}-\mathbf{x_1}}{|\mathbf{x_1}-\mathbf{x_2}|}, \quad \mathbf{n} \equiv \frac{\mathbf{x_1}-\mathbf{x_2}}{|\mathbf{x_1}-\mathbf{x_2}|}$

$$\Rightarrow \quad 16\pi^2\epsilon_0 W_{\text{int}} = q_1 q_2 \int \frac{(\mathbf{x}-\mathbf{x_1})\cdot(\mathbf{x}-\mathbf{x_2})}{|\mathbf{x}-\mathbf{x_1}|^3|\mathbf{x}-\mathbf{x_2}|^3}\,\mathrm{d}^3x = \frac{q_1 q_2}{|\mathbf{x_1}-\mathbf{x_2}|}\int \frac{\boldsymbol{\rho}\cdot(\boldsymbol{\rho}+\mathbf{n})}{\rho^3|\boldsymbol{\rho}+\mathbf{n}|^3}\,\mathrm{d}^3\rho$$

$$= \frac{q_1 q_2}{|\mathbf{x_1}-\mathbf{x_2}|}\int \nabla\rho^{-1}\cdot\nabla|\boldsymbol{\rho}+\mathbf{n}|^{-1}\,\mathrm{d}^3\rho = -\frac{q_1 q_2}{|\mathbf{x_1}-\mathbf{x_2}|}\int \frac{\nabla^2\rho^{-1}}{|\boldsymbol{\rho}+\mathbf{n}|}\,\mathrm{d}^3\rho$$

$$\Rightarrow \quad W_{\text{int}} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\mathbf{x_1}-\mathbf{x_2}|}\int \frac{\delta^3(\rho)}{|\boldsymbol{\rho}+\mathbf{n}|}\,\mathrm{d}^3\rho = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\mathbf{x_1}-\mathbf{x_2}|} \quad \Leftarrow \quad \text{as expected}$$

- Forces acting between charged bodies can be obtained by calculating the change in the total electrostatic energy of the system under small virtual displacements.

- To calculate the force per unit area on the surface of a conductor with a surface-charge density, imagine a small outward displacement of an elemental area of the surface

$$w = \frac{\epsilon_0}{2}|\mathbf{E}|^2 = \frac{\sigma^2}{2\epsilon_0} \quad \Rightarrow \quad \Delta W = -\frac{\sigma^2}{2\epsilon_0}\Delta a \Delta x$$

an outward force per unit area equal to w at the surface of the conductor.

- For a system of n conductors, the electrostatic potential energy can be expressed in terms of the potentials alone and certain geometrical quantities called coefficients of capacity

$$V_i = \sum_{j=1}^{n} p_{ij} Q_j \qquad (i = 1, 2, \cdots, n) \quad \Rightarrow \quad Q_i = \sum_{j=1}^{n} C_{ij} V_j \qquad (i = 1, 2, \cdots, n)$$

C_{ii} : capacities or capacitances , $C_{ij}\,(i \neq j)$: coefficients of induction

- The capacitance of a conductor is the total charge on the conductor when it is maintained at unit potential, all other conductors being held at zero potential.

- the capacitance of 2 conductors carrying equal and opposite charges in the presence of other grounded conductors is defined as the ratio of the charge on one conductor to the potential difference between them.

- The potential energy for the system of conductors $W = \frac{1}{2}\sum_{i=1}^{n} Q_i V_i = \frac{1}{2}\sum_{i,j=1}^{n} C_{ij} V_i V_j$

- The expression of the energy in terms of the potentials and the C_{ij}, or in terms of the charges and the coefficients p_{ij} permits the application of variational methods to obtain approximate values of capacitances.

- Selected problems: 1.3, 1.6, 1.9, 1.10, 1.14, 1.15, 1.17, 1.21

1.12 Variational Approach to the Solution of the Laplace and Poisson Equations

- Variational methods provide formal techniques for the derivation of "equations of motion" and also practical methods for obtaining approximate, but often accurate, solutions to problems not amenable to other approaches.

- consider the functional $I[\psi] = \frac{1}{2}\int_V (\nabla\psi\cdot\nabla\psi - 2\,g\,\psi)\,\mathrm{d}^3x \quad \Leftarrow g(\mathbf{x})$: source function

$$\delta I = I[\psi+\delta\psi] - I[\psi] = \int_V [\nabla\psi\cdot\nabla\delta\psi + \frac{1}{2}(\nabla\delta\psi)^2 - g\,\delta\psi]\,\mathrm{d}^3x$$

$$\approx \int_V [-\nabla^2\psi - g]\,\delta\psi\,\mathrm{d}^3x + \oint_S \delta\psi\,\frac{\partial\psi}{\partial n}\,\mathrm{d}a \quad \Leftarrow \quad \begin{matrix}\phi = \delta\psi \\ \psi = \psi\end{matrix} \quad \text{in Green's 1st identity}$$

$$\delta\psi\big|_S = 0 \quad \& \quad \nabla^2\psi + g = 0 \quad \Rightarrow \quad \delta I^1 \equiv \text{1st-order of} \quad \delta I = 0$$

- I is a stationary minimum if ψ satisfies a Poisson-like equation within the volume and the departures $\delta\psi$ vanish on the boundary.

- the minimization of the functional yields the "eqn of motion" of the electrostatic potential in the presence of a charge density and Dirichlet boundary conditions.

- the stationary nature of the extremum of I permits a practical approach to an approximate solution for $\psi(\mathbf{x})$.

- a: choose a flexible "trial" function $\psi(\mathbf{x}) = A\Psi(\mathbf{x},\alpha,\beta,\ldots)$ to satisfy the boundary conditions on the surface.

  b: calculation of $I[\psi]$ gives $I(A,\alpha,\beta,\ldots)$.

  c: vary the parameters to locate the extremum of $I(A,\alpha,\beta,\ldots)$.

- With the optimum parameters, the trial solution is the best possible approximation to the true solution with the particular functional form chosen.

- For the Laplace eqn, the normalization constant A is determined by the Dirichlet boundary condition. For the Poisson eqn, it is determined by the source strength $g(\mathbf{x})$, as well as the boundary values.

- For Neumann boundary conditions

$$I[\psi] = \int_V \left(\frac{1}{2}\nabla\psi\cdot\nabla\psi - g\,\psi\right) \mathrm{d}^3x - \oint_S f\,\psi\,\mathrm{d}\,a \quad \Leftarrow \quad f(\mathbf{s}) \text{ is defined on } S$$

$$\Rightarrow \quad \delta I^1 = \int_V \left[-\nabla^2\psi - g\right]\delta\psi\,\mathrm{d}^3x + \oint_S \left[\frac{\partial\psi}{\partial n} - f(\mathbf{x})\right]\delta\psi\,\mathrm{d}\,a$$

$$\nabla^2\psi + g = 0 \;\;\&\;\; \left.\frac{\partial\psi}{\partial n}\right|_S = f(\mathbf{s}) \quad \Rightarrow \quad \delta I^1 = 0 \text{ independent of } \delta\psi$$

- Approximate solutions can be found by using trial functions that satisfy the Neumann boundary conditions.

- consider a hollow circular cylinder of unit radius centered on the z-axis with an interior source density $g(\mathbf{x})=g(\rho)$, $\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)=-g(\rho) \quad \& \quad \psi(\rho=1)=0$

- Two possible trial functions:
$$\Psi_1=\alpha_1(1-\rho)+\beta_1(1-\rho)^2+\gamma_1(1-\rho)^3$$
$$\Psi_2=\alpha(\rho^2-1)+\beta(\rho^3-1)+\gamma(\rho^4-1)$$

- Ψ_1 is a less accurate representation of ψ because ψ should have a maximum or minimum at the origin with vanishing slope.

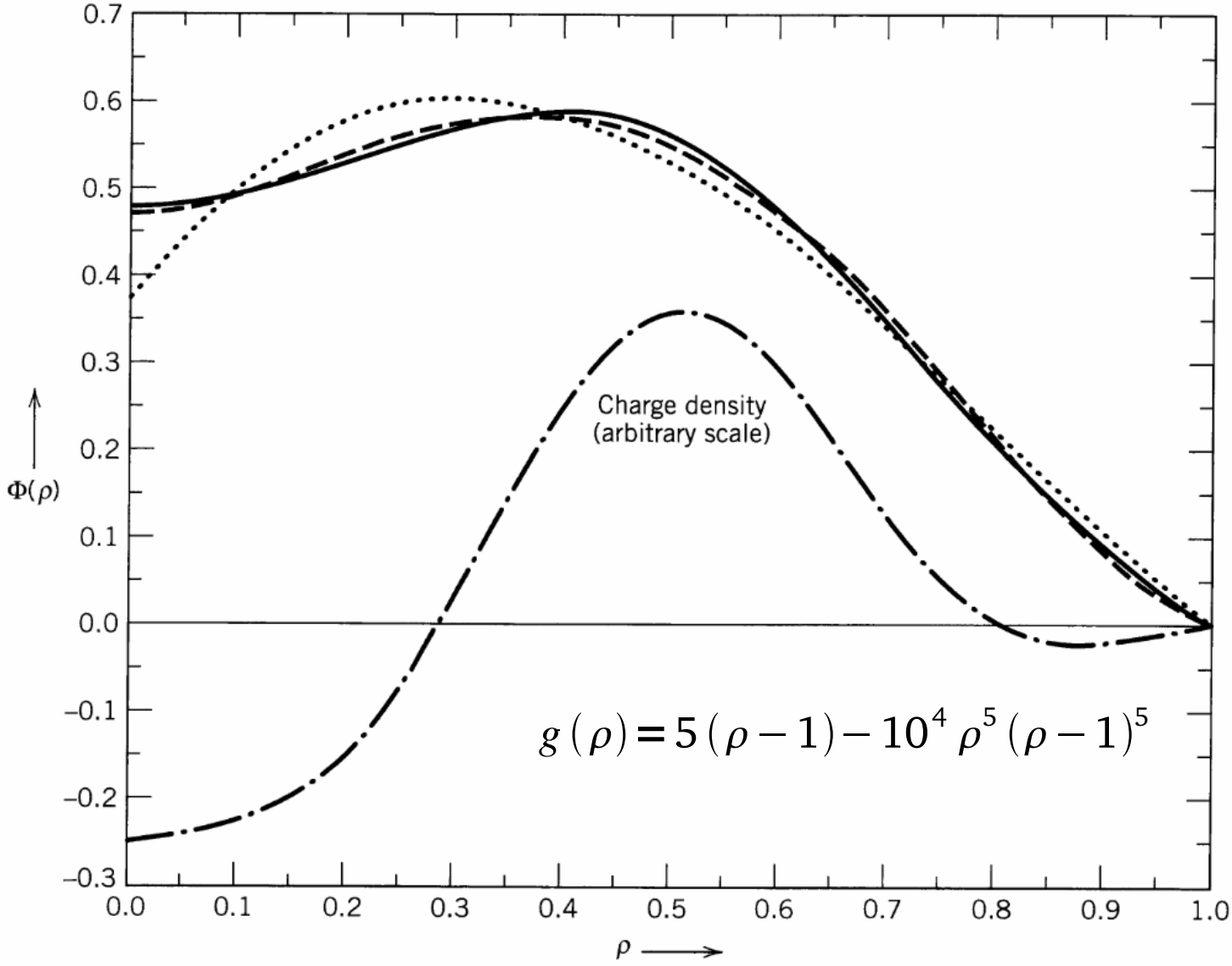
- $$\frac{1}{2\pi}I[\Psi_2]=\frac{1}{2}\alpha^2+\frac{6}{5}\alpha\beta+\frac{4}{3}\alpha\gamma+\frac{3}{4}\beta^2+\frac{12}{7}\beta\gamma+\gamma^2-e_2\alpha-e_3\beta-e_4\gamma$$

where $e_n=\int_0^1 g(\rho)(\rho^n-1)\rho\,d\rho$

$$\frac{\partial I}{\partial\alpha}=0 \qquad \alpha=225\,e_2-420\,e_3+210\,e_4$$
$$\frac{\partial I}{\partial\beta}=0 \quad \Rightarrow \quad \beta=-420\,e_2+\frac{2450}{3}e_3-420\,e_4$$
$$\frac{\partial I}{\partial\gamma}=0 \qquad \gamma=210\,e_2-420\,e_3+\frac{441}{2}e_4$$

- Ψ_1 fails badly for $\rho<0.3$ because it does not respect the vanishing slope at $\rho=0$.

- The insensitivity of $I[\Psi]$ to errors in the trial function shows both a strength and a weakness of the variational method. If the principle is used to estimate eigenvalues, it does well. Used as a method of estimating a solution, it can fail badly, at least in parts of the configuration space.

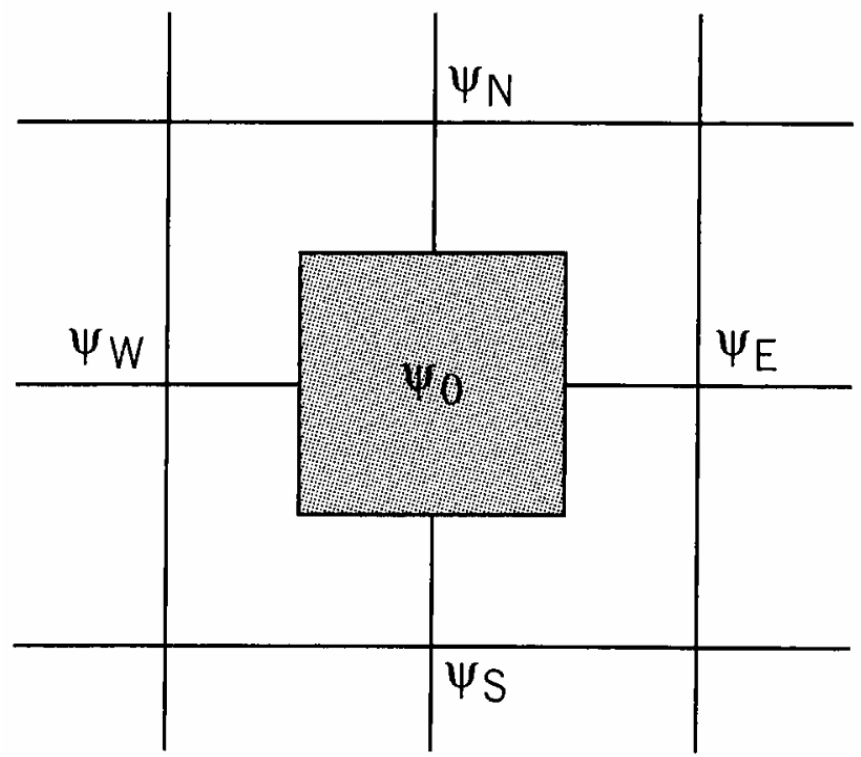


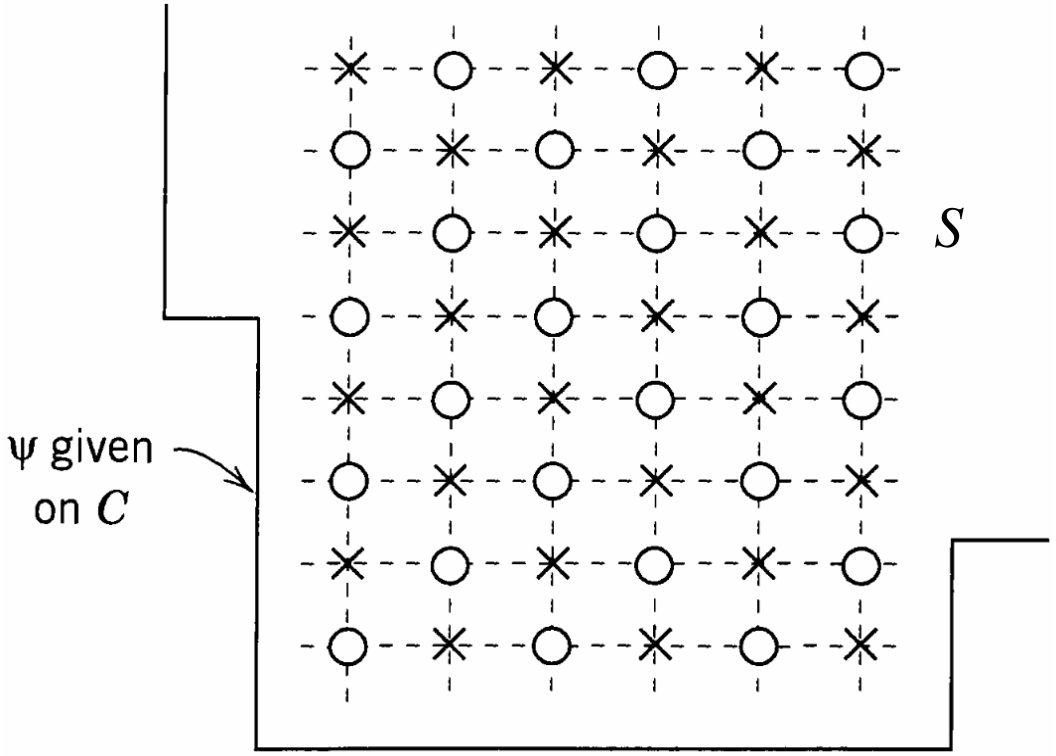
1.13 Relaxation Method for 2D Electrostatic Problems

- The relaxation method (iterative finite difference method) is an iterative numerical scheme for the solution of the Laplace or Poisson equation.

- First consider the Laplace equation with Dirichlet boundary conditions within a region with a boundary contour. The region is spanned by a square lattice with lattice spacing h. The potential values on the boundary sites are assumed given.

- imagine the functional integral $I[\psi]$ over S as a sum over small domains. Consider the neighboring trial values of the potential as fixed, while the value at the center of the sub-area is a variational quantity to be optimized.





- approximate the derivatives in the northeast quarter

$$\left(\frac{\partial \psi}{\partial x}\right)_{\mathrm{NE}} \approx \frac{\psi_{\mathrm{E}} - \psi_0}{h}, \quad \left(\frac{\partial \psi}{\partial y}\right)_{\mathrm{NE}} \approx \frac{\psi_{\mathrm{N}} - \psi_0}{h}$$ similarly for the other three quarters

$$\Rightarrow \quad I_{\mathrm{NE}} = \frac{1}{2}\int_0^{h/2} \mathrm{d}\, x \int_0^{h/2} \mathrm{d}\, y \left[\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2\right] \approx \frac{1}{8}\left[(\psi_{\mathrm{E}} - \psi_0)^2 + (\phi_{\mathrm{N}} - \psi_0)^2\right] \qquad (**)$$

$$\Rightarrow \quad I \approx \frac{1}{4}\left[(\psi_{\mathrm{E}} - \psi_0)^2 + (\psi_{\mathrm{S}} - \psi_0)^2 + (\psi_{\mathrm{W}} - \psi_0)^2 + (\phi_{\mathrm{N}} - \psi_0)^2\right]$$

$$\Rightarrow \quad (\psi_0)_{\text{optimum}} = \frac{1}{4}(\psi_{\mathrm{E}} + \psi_{\mathrm{S}} + \psi_{\mathrm{W}} + \psi_{\mathrm{N}}) \quad \Leftarrow$$

Minimizing I with respect to ψ_0 gives the optimum value

- I is minimized if ψ_0 is equal to the average of the values at the "cross" points.

- consider the whole functional integral. guess a set of $\psi(i,j)$ initially and approximate $I[\psi]$ by the sum of terms of the form of (**). go over the lattice and replace half the values by the average of the points (crosses) around them.

- Repeat the process to do the other half points.

- We could take the average of the values at the corners of the large square instead of the "cross" values, or we could take some linear combination of the two to improve its accuracy.

- Choose $\langle\langle F(x,y)\rangle\rangle \equiv \frac{4}{5}\langle F\rangle_c + \frac{1}{5}\langle F\rangle_s$

$$\langle F\rangle_c = \frac{1}{4}\left[F(x+h,y)+F(x,y+h)+F(x-h,y)+F(x,y-h)\right]$$

where

$$\langle F\rangle_s = \frac{1}{4}\left[F(x+h,y+h)+F(x+h,y-h)+F(x-h,y+h)+F(x-h,y-h)\right]$$

$$\Rightarrow \quad \langle\langle F(x,y)\rangle\rangle = F(x,y)+\frac{3}{10}h^2\nabla^2 F+\frac{1}{40}h^4\nabla^2\nabla^2 F+O(h^6) \quad \Leftarrow \quad \text{problem 1.22}$$

$$\Rightarrow \quad \psi_{\text{new}}(i,j) = \langle\langle\psi(i,j)\rangle\rangle + O(h^6) \quad \Leftarrow \quad \nabla^2\psi = 0 \quad (\text{Lapalce eqn})$$

- With either the "cross" or "square" averaging separately, the error is $O(h^4)$.

- The increase in accuracy is at the expense of twice as much computation for each lattice site, but for the same accuracy, far fewer lattice sites are needed

$$\langle\langle N\rangle\rangle = O\left(\langle N\rangle^{2/3}\right) \quad \Leftarrow \quad \begin{array}{l}\langle\langle N\rangle\rangle : \text{the number of sites needed with } \langle\langle\psi\rangle\rangle \\ \langle N\rangle \ : \text{the number with the cross or square average}\end{array}$$

- For the Poisson equation: $\nabla^2\psi = -g$

$$\psi_{\text{new}}(i,j) = \langle\langle\psi(i,j)\rangle\rangle + \frac{h^2}{5}g(i,j) + \frac{h^2}{10}\langle g(i,j)\rangle_c + O(h^6)$$

- A basic procedure (Jacobian iteration) for the iterative numerical solution of the Laplace or Poisson equation with Dirichlet boundary conditions

1: A square lattice spacing is chosen and the lattice sites, including the sites on the boundary.

2: The values of the potential at the boundary sites are entered in a table of the potential at all sites.

3: A guess is made for the values, $\Phi_{old}(i,j)$, at all interior sites.

4: The 1st iteration cycle begins by systematically going over the lattice sites and computing $\langle\langle\Phi(i,j)\rangle\rangle$ or any of the averages. This quantity is entered as $\Phi_{new}(i,j)$ in a table of "new" values of the potential at each site.

5: replace the set of Φ_{old} by the set of Φ_{new} and the iteration cycle begins again.

6: Iterations continue until some desired level of accuracy is achieved.

- The Gauss-Seidel iteration, a better one, replaces Φ_{old} with Φ_{new} as soon as the latter is determined.

- consult *Numerical Recipes* for other possible improvements.