# Mansuripur's Paradox

Kirk T. McDonald
*Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544*
(May 2, 2012)

## 1 Problem

An electrically neutral current-loop, with magnetic dipole $\mathbf{m}_0$, that is at rest in a static, uniform electric field $\mathbf{E}$ experiences no force or torque. However, if that system is observed in the lab frame where the loop has velocity $\mathbf{v}$ parallel to $\mathbf{E}$, and $v \ll c$, where $c$ is the speed of light, then there appears to be an electric dipole moment,[1] $\mathbf{p} = \mathbf{v}/c \times \mathbf{m}_0$ associated with the loop (in Gaussian units). The torque on this moment due to the electric field (which has strength $\mathbf{E} + \mathcal{O}(v^2/c^2)$ in this frame) is[2] $\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E} = Ev\mathbf{m}_0/c$.

Can/should the torque be different in different frames of reference?

The paradox is compounded by supposing the static, "uniform" electric field is due to a single electric charge $q$ at large, fixed distance from the magnetic moment in the rest frame of the latter, and the lab-frame velocity $\mathbf{v}$ is along the line of centers of the charge and moment. Discuss the force on charge $q$ in the lab frame.

*This paradox was recently posed by Mansuripur [2]. It is a conceptual variant of a famous problem by Shockley [3] that introduced the concept of "hidden mechanical momentum."*

## 2 Solution

Note that the magnetic moment of a loop of current $I$ of radius $a$ has the magnitude $m_0 = \pi a^2 I/c$, so the lab-frame torque, of magnitude $\tau = Evm_0/c = \pi a^2 IEv/c^2$, is an effect of order $1/c^2$. Hence, the analysis of the problem should include all effects at order $1/c^2$.

### 2.1 Polarization Precession?

The presence of the torque $\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}$ suggests that the electric dipole moment $\mathbf{p}$ would "precess" about $\mathbf{m}_0$ so as to bring it into alignment with the electric field $\mathbf{E}$.

However, the apparent electric moment $\mathbf{p} = \mathbf{v}/c \times \mathbf{m}_0$ is independent of $\mathbf{E}$, and so cannot be expected to move into alignment with that field.[3]

It must be that the torque has no effect on the mechanical configuration of the system.

[1] See, for example, eq. (2) of [1].

[2] See the Appendix.

[3] When $\mathbf{v}$ is parallel to $\mathbf{E}$ (and $\mathbf{B} = 0$), there is no precession of the magnetic moment $\mathbf{m}_0$. See, for example, [4, 5].

## 2.2 Field Momentum and "Hidden" Mechanical Momentum in the Rest Frame

Before considering the problem further in the lab frame, it is useful to note a subtlety in the rest frame of the system. Namely, the system at rest possesses nonzero electromagnetic field momentum $\mathbf{P}_{\rm EM}$. Since a system at rest must have zero total momentum, it must also possess a "hidden" mechanical momentum $\mathbf{P}_{\rm mech}$ equal and opposite to the field momentum.[4] This "hidden" momentum is a relativistic effect, of order $1/c^2$.

For systems in which effects of radiation and of retardation can be ignored, the electromagnetic momentum can be calculated in various equivalent ways [7],

$$\mathbf{P}_{\rm EM} = \int \frac{\varrho \mathbf{A}}{c}\, d{\rm Vol} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} = \int \frac{V\mathbf{J}}{c^2}\, d{\rm Vol}, \tag{1}$$

where $\varrho$ is the electric charge density, $\mathbf{A}$ is the magnetic vector potential (in the Coulomb gauge where $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$), $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, $V$ is the electric (scalar) potential, and $\mathbf{J}$ is the electric current density. The first form is due to Faraday [8] and Maxwell [9], the second form is due to Poynting [11] and Abraham [10], and the third form was introduced by Furry [12].

We evaluate $\mathbf{P}_{\rm EM}$ for a magnetic moment $\mathbf{m} = \pi a^2 I\,\hat{\mathbf{z}}/c$ due to current $I$ which flows in a circular loop of radius $a$ subject to external electric field $\mathbf{E}$ that makes angle $\alpha$ to $\mathbf{m}$, *i.e.*, $\mathbf{E} = E(\sin\alpha\,\hat{\mathbf{x}} + \cos\alpha\,\hat{\mathbf{z}})$. The largest magnetic field is inside the loop, in the $\hat{\mathbf{z}}$ direction, so the second form of eq. (1) indicates that $\mathbf{P}_{\rm EM}$ will be in the $-\hat{\mathbf{y}}$ direction. This result is counterintuitive in that the direction of the momentum is not related to the direction of the velocity (if any). In the present problem $\mathbf{P}_{\rm EM}$ is perpendicular to $\mathbf{v}$, so the field angular momentum (5) is nonzero and position/time dependent for motion along the $x$-axis.

We use the third form of eq. (1) to compute the field momentum. The external electric field can be derived from the scalar potential $V = -E(x\sin\alpha + z\cos\alpha)$,[5] and the $y$-component of $\mathbf{J}\,d{\rm Vol}$ is $I a\cos\phi\, d\phi$ in cylindrical coordinates $(\rho,\phi,z)$ centered on the moment. Then, noting that $x = a\cos\phi$ and $z = 0$ on the loop, we find

$$P_{{\rm EM},y} = \int \frac{V J_y}{c^2}\, d{\rm Vol} = \int_0^{2\pi} \frac{(-Ea\cos\phi\sin\alpha)(Ia\cos\phi)}{c^2}\, d\phi = -\frac{\pi a^2 I E\sin\alpha}{c^2} = -\frac{mE\sin\alpha}{c}\,. \tag{2}$$

That is,[6]

$$\mathbf{P}_{\rm EM} = \frac{\mathbf{E}\times\mathbf{m}}{c}\,. \tag{3}$$

As the total momentum of the system at rest must be zero, we infer that there exists "hidden" mechanical momentum given by

$$\mathbf{P}_{\rm mech} = -\mathbf{P}_{\rm EM} = -\frac{\mathbf{E}\times\mathbf{m}}{c}\,. \tag{4}$$

The momenta (3)-(4) are effects of order $1/c^2$.

[4] For commentary on "hidden" momentum, see [6].

[5] In principle, we should also consider the scalar potential associated with the electric dipole $\mathbf{p} = \mathbf{v}/c\times\mathbf{m}_0$, but this leads to a contribution to the torque of order $v^2/c^2$, which we neglect.

[6] By a similar calculation [13], the field momentum of an electric dipole $\mathbf{p}$ in a transverse magnetic field $\mathbf{B}$ is $\mathbf{P}_{\rm EM} = \mathbf{B}\times\mathbf{p}/2c$.

## 2.3 Torque and Changing "Hidden" Angular Momentum

A classical magnetic moment $\mathbf{m}_0$ has intrinsic mechanical angular momentum $\mathbf{L}_0 = 2Mc\,\mathbf{m}_0/Q$ where $M$ and $Q$ are the mass and charge of the particles whose motion generates the moment. In addition, the moment is associated with "hidden" mechanical angular momentum given by

$$\mathbf{L}_{\rm hidden} = \mathbf{r} \times \mathbf{P}_{\rm mech}, \tag{5}$$

where $\mathbf{r}$ is the position of the center of the moment.

In the inertial frame where the magnetic moment has position $\mathbf{r} = \mathbf{v}t = vt\,\hat{\mathbf{x}}$, with $v \ll c$ (such that the electric field and the moment have the same values as in the moment's rest frame to order $v/c$, and the field momentum and the "hidden" mechanical momentum have their rest-frame values to order $1/c^2$), the mechanical angular momentum of the system is[7]

$$\mathbf{L}_{\rm mech} = \mathbf{L}_0 + \mathbf{L}_{\rm hidden} = \mathbf{L}_0 - \mathbf{v}t \times \frac{\mathbf{E} \times \mathbf{m}_0}{c}\,. \tag{6}$$

To support this time-varying mechanical angular momentum, the system must be subject to a torque,[8]

$$\boldsymbol{\tau} = \frac{d\mathbf{L}_{\rm mech}}{dt} = -\mathbf{v} \times \frac{\mathbf{E} \times \mathbf{m}_0}{c}\,. \tag{7}$$

When $\mathbf{E}$ and $\mathbf{v}$ are parallel, we can rewrite eq. (7) as

$$\boldsymbol{\tau} = -\mathbf{E} \times \frac{\mathbf{v} \times \mathbf{m}_0}{c} = \mathbf{p} \times \mathbf{E}. \tag{8}$$

That is, the "paradoxical" nonzero torque is needed to change the "hidden" mechanical angular momentum of the system, such that this remains equal and opposite to the field angular momentum, which latter appears to be time dependent in the lab frame.

## 2.4 Physical Realizations of Magnetic Moments

The behavior of a moving current loop in an external electric field depends on the physical nature of the current.

If the current flows in a resistive conductor, that conductor would "shield" the current from a constant, uniform external electric field $\mathbf{E}$ if the conductor is at rest or in uniform motion with respect to the field. In this case there would be no Lorentz force on the current due to the external field, and no torque in the frame where the current loop has velocity $\mathbf{v}$.

Similarly, if the current loop is a superconductor, the supercurrent is "shielded" from the external field, and there is no torque.

A model of a neutral current loop that could realize Mansuripur's paradox is a pair of nonconducting, coaxial disks with positive charge fixed to the rim of one and negative charge

[7]The intrinsic mechanical angular momentum $\mathbf{L}_0$ has corrections at order $v^2/c^2$, but there are time-independent in the lab frame.

[8]Hence, it was wrong of Mansuripur [2] to claim that the existence of a nonzero torque on a moving magnetic moment is inconsistent with special relativity.

on the other, with the disks rotating in opposite senses with the same magnitude of angular velocity. The paradox applies also to models in which the current is a charged, compressible gas or liquid that flow inside a nonconducting tube (models i and iii of [17]).[9]

### 2.5 The External Field is Due to a Single Distant Charge

Another paradox arises if we suppose that the external field $\mathbf{E} = E, \hat{\mathbf{x}}$ is due to a single charge $q$ at $x = -d_0$ for large $d$.

In the lab frame we might argue that the force on $q$ is due to both the electric field from the apparent electric dipole $\mathbf{v}/c \times \mathbf{m}_0$ and the magnetic field of the magnetic moment $\mathbf{m}_0$,

$$\mathbf{F}_q = q\left(\mathbf{E}_p + \frac{\mathbf{v}}{c} \times \mathbf{B}_m\right) = q\left(-\frac{\mathbf{p}}{d^3} + \frac{\mathbf{v}}{c} \times \frac{-\mathbf{m_0}}{d^3}\right) = -2q\frac{\mathbf{v}}{c} \times \frac{\mathbf{m_0}}{d^3} = -\frac{2qvm_0}{d^3}\,\hat{\mathbf{y}}. \tag{9}$$

But, the force on the magnetic moment is zero in the lab frame. How can this be?

The issue is that the fields of a moving dipole are not the same as the fields of the dipoles obtained by the transformation of the moments in their rest frame [1]. That is, the meaning of a moving dipole must be considered with care.

The proper calculation is that

$$\mathbf{F}_q = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right), \tag{10}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the Lorentz transformations of the fields of the magnetic moment $\mathbf{m}_0$ in its rest frame, where

$$\mathbf{E}_0 = 0, \qquad \mathbf{B}_0 = -\frac{\mathbf{m_0}}{d^3} \tag{11}$$

at charge $q$. The transforms of these to the lab frame are

$$\mathbf{E} = \mathbf{E}_0 - \frac{\mathbf{v}}{c} \times \mathbf{B}_0 = -\frac{\mathbf{v}}{c} \times \mathbf{B}_0, \qquad \mathbf{B} = \mathbf{B}_0 + \frac{\mathbf{v}}{c} \times \mathbf{E}_0 = \mathbf{B}_0. \tag{12}$$

Using these in eq. (10) we find $\mathbf{F}_q = 0$ as expected.

It remains disconcerting that the electric field in the lab frame at charge $q$ is the negative of that inferred from the relation $\mathbf{p} = \mathbf{v}/c \times \mathbf{m}_0$.

# A Appendix

*This Appendix transcribes certain arguments of Namias [15]. For related discussion, see [16, 17].*

The expression $\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}$ is valid for the torque on an electric dipole that is at rest in a static electric field, but if a magnetic field $\mathbf{B}$ is present the torque on an electric dipole $\mathbf{p} = q(\mathbf{r}_+ - \mathbf{r}_-)$ with velocity $\mathbf{v}$ is given by

$$\begin{aligned}\boldsymbol{\tau}_p &= \mathbf{r}_+ \times q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) + \mathbf{r}_- \times -q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) = q(\mathbf{r}_+ - \mathbf{r}_-) \times \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \\ &= \mathbf{p} \times \mathbf{E} + \mathbf{p} \times \left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right),\end{aligned} \tag{13}$$

[9] To have an electrically neutral current loop, one must postulate a pair of such tubes that containing opposite charged gas/liquid flowing in opposite directions.

in the limit of a point dipole. Similarly, the torque on a moving, point magnetic dipole $\mathbf{m}$ due to external fields can be deduced by supposing that the dipole consists of a pair of magnetic charges $\pm q_M$ subject to the Lorentz force $q_M(\mathbf{B} - \mathbf{v}/c \times \mathbf{E})$, which leads to

$$\begin{aligned} \boldsymbol{\tau}_m &= \mathbf{r}_+ \times q_M \left(\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}\right) + \mathbf{r}_- \times -q_M \left(\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}\right) = q_M(\mathbf{r}_+ - \mathbf{r}_-) \times \left(\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}\right) \\ &= \mathbf{m} \times \mathbf{B} - \mathbf{m} \times \left(\frac{\mathbf{v}}{c} \times \mathbf{E}\right). \end{aligned} \tag{14}$$

In the present example, the external electric field in the frame in which the magnetic dipole has velocity $\mathbf{v}$, with $v \ll c$, is just $\mathbf{E}$ to order $v/c$, and the external magnetic field is $\mathbf{B} = -\mathbf{v}/c \times \mathbf{E}$. Also, the magnetic moment is $\mathbf{m}_0$ and the electric dipole moment is $\mathbf{p} = \mathbf{v}/c \times \mathbf{m}_0$ in this frame, to order $v/c$. Then, to this order, the total torque on the moving dipole is

$$\boldsymbol{\tau} = \boldsymbol{\tau}_p + \boldsymbol{\tau}_m = \mathbf{p} \times \mathbf{E} - \mathbf{m}_0 \times \left(\frac{\mathbf{v}}{c} \times \mathbf{E}\right). \tag{15}$$

While the torque is not equal to $\mathbf{p} \times \mathbf{E}$ in general, it does equal this if $\mathbf{v}$ is parallel to $\mathbf{E}$ (or if $\mathbf{m}_0$ is parallel to $\mathbf{v} \times \mathbf{E}$).

Spin-1/2 elementary particles have non-classical magnetic moments. As was noted by Fermi [14], the behavior of these moments at the origin in hyperfine interactions indicates that they are the quantum equivalents of current loops, rather than pairs of equal and opposite magnetic charges. It is well-known that these intrinsic moments do not precess when they move in an electric field with velocity $\mathbf{v}$ parallel to an external electric field $\mathbf{E}$ (see, for example, [4, 5].)

# References


[1] K.T. McDonald, *Fields and Moments of a Moving Electric Dipole* (Nov. 29, 2011), http://puhep1.princeton.edu/~mcdonald/examples/movingdipole.pdf

[2] M. Mansuripur, *Trouble with the Lorentz law of force: Incompatibility with special relativity and momentum conservation* (May 1, 2012), http://arxiv.org/ftp/arxiv/papers/1205/1205.0096.pdf

[3] W. Shockley and R.P. James, *"Try Simplest Cases" Discovery of "Hidden Momentum" Forces on "Magnetic Currents"*, Phys. Rev. Lett. **18**, 876 (1967), http://puhep1.princeton.edu/~mcdonald/examples/EM/shockley_prl_18_876_67.pdf

[4] V. Bargmann, L. Michel and V.L. Telegdi, *Precession of the Polarization of Particles Moving in a Homogeneous Electromagnetic Field*, Phys. Rev. Lett. **2**, 435 (1959), http://puhep1.princeton.edu/~mcdonald/examples/QED/bargmann_prl_2_435_59.pdf

[5] K.T. McDonald, *Polarization Precession* (Jan. 14, 1970), http://puhep1.princeton.edu/~mcdonald/examples/polprecess.pdf

[6] K.T. McDonald, *"Hidden" Momentum of a Steady Current Distribution in a System at "Rest"* (Apr. 21, 2009),
http://puhep1.princeton.edu/~mcdonald/examples/current.pdf

[7] J.D. Jackson, *Relation between Interaction terms in Electromagnetic Momentum* $\int d^3\mathbf{x}\,\mathbf{E}\times\mathbf{B}/4\pi c$ *and Maxwell's* $e\mathbf{A}(\mathbf{x},t)/c$*, and Interaction terms of the Field Lagrangian* $\mathcal{L}_{\rm em} = \int d^3\mathbf{x}\,[E^2 - B^2]/8\pi$ *and the Particle Interaction Lagrangian,* $\mathcal{L}_{\rm int} = e\phi - e\mathbf{v}\cdot\mathbf{A}/c$ (May 8, 2006), http://puhep1.princeton.edu/~mcdonald/examples/EM/jackson_050806.pdf

[8] Part I, sec. IX of M. Faraday, *Experimental Researches in Electricity* (Dover Publications, New York, 2004; reprint of the 1839 edition).

[9] Secs. 22-24 and 57 of J.C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **155**, 459 (1865),
http://puhep1.princeton.edu/~mcdonald/examples/EM/maxwell_ptrsl_155_459_65.pdf

[10] M. Abraham, *Prinzipien der Dynamik des Elektrons*, Ann. Phys. **10**, 105 (1903),
http://puhep1.princeton.edu/~mcdonald/examples/EM/abraham_ap_10_105_03.pdf

[11] J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **175**, 343 (1884),
http://puhep1.princeton.edu/~mcdonald/examples/EM/poynting_ptrsl_175_343_84.pdf

[12] W.H. Furry, *Examples of Momentum Distributions in the Electromagnetic Field and in Matter*, Am. J. Phys. **37**, 621 (1969),
http://puhep1.princeton.edu/~mcdonald/examples/EM/furry_ajp_37_621_69.pdf

[13] K.T. McDonald, *Electromagnetic Momentum of a Capacitor in a Uniform Magnetic Field* (June 18, 2006), http://puhep1.princeton.edu/~mcdonald/examples/cap_momentum.pdf

[14] E. Fermi, *Über die magnetischen Momente der Atomkerne*, Z. Phys. **60**, 320 (1930),
http://puhep1.princeton.edu/~mcdonald/examples/QED/fermi_zp_60_320_30.pdf

[15] V. Namias, *Electrodynamics of moving dipoles: The case of the missing torque*, Am. J. Phys. **57**, 171 (1989),
http://puhep1.princeton.edu/~mcdonald/examples/EM/namias_ajp_57_171_89.pdf

[16] D. Bedford and P. Krumm, *On the origin of magnetic dynamics*, Am. J. Phys. **54**, 1036 (1986), http://puhep1.princeton.edu/~mcdonald/examples/EM/bedford_ajp_54_1036_86.pdf

[17] l. Vaidman, *Torque and force on a magnetic dipole*, Am. J. Phys. **58**, 978 (1990),
http://puhep1.princeton.edu/~mcdonald/examples/EM/vaidman_ajp_58_978_90.pdf